simulation snapshot: O.Hahn & R.Angulo

Shell-crossing & the origin of cosmic structures

Lagrange seminar, 07/07/2020

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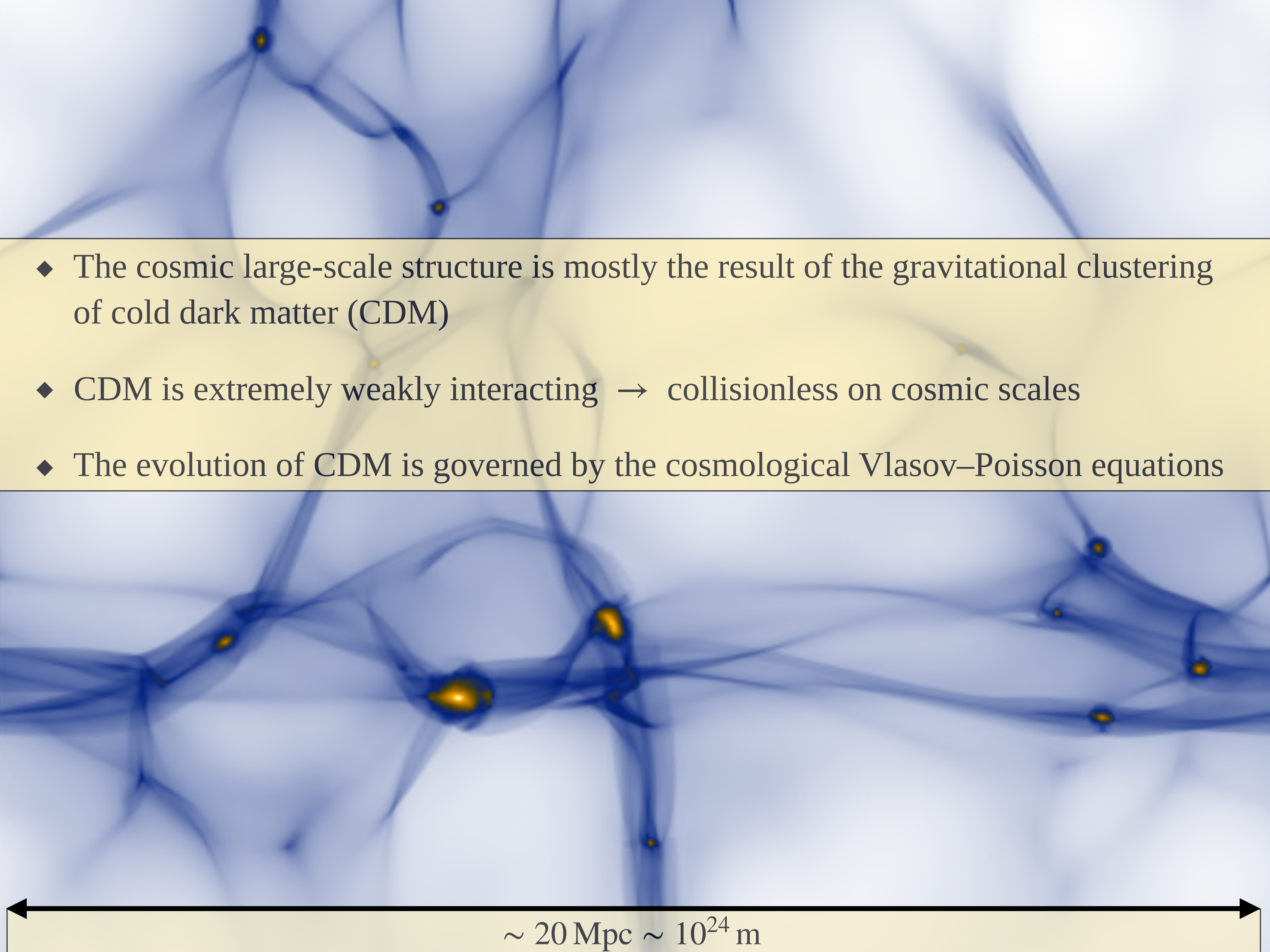
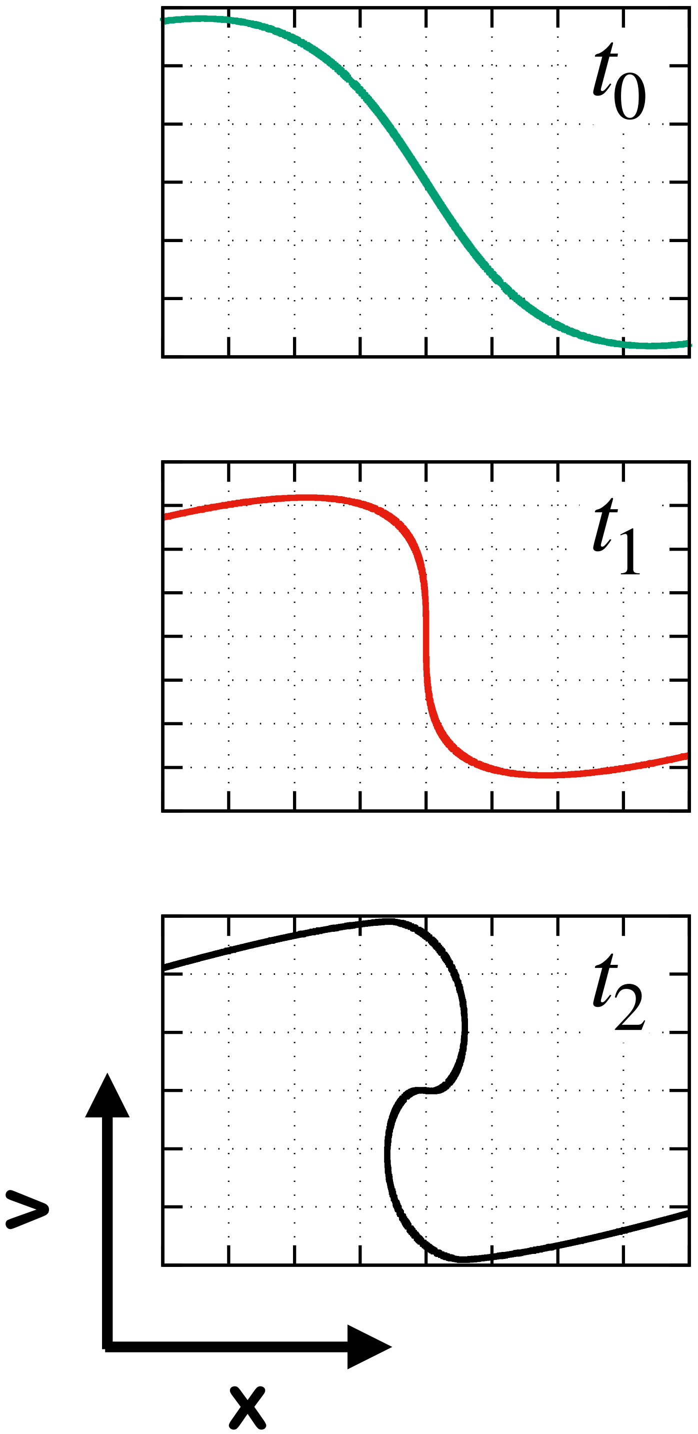
MARIE CURIE ACTIONS



- The cosmic large-scale structure is mostly the result of the gravitational clustering of cold dark matter (CDM)
- CDM is extremely weakly interacting → collisionless on cosmic scales
- The evolution of CDM is governed by the cosmological Vlasov–Poisson equations

CDM phase-space:

$(t_0 < t_1 < t_2)$



at early times CDM is in the **single-stream regime**
(= comes with single-valued velocity)

collisionless nature of CDM leads to
crossing of trajectories, called **shell-crossing**

after that, CDM is in the **multi-stream regime**
(with non-zero velocity dispersion)

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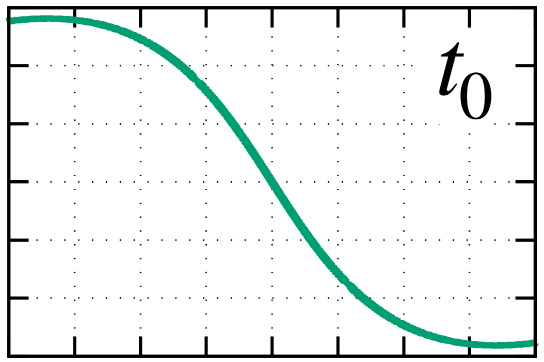

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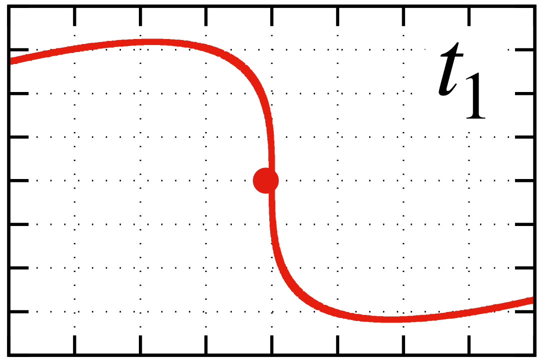
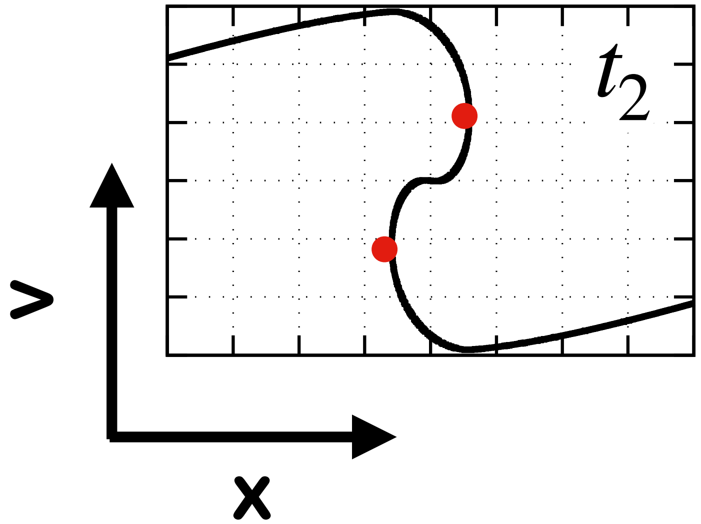


collisionless nature of CDM leads to crossing of trajectories, called **shell-crossing**
(density at ● $\to \infty$)



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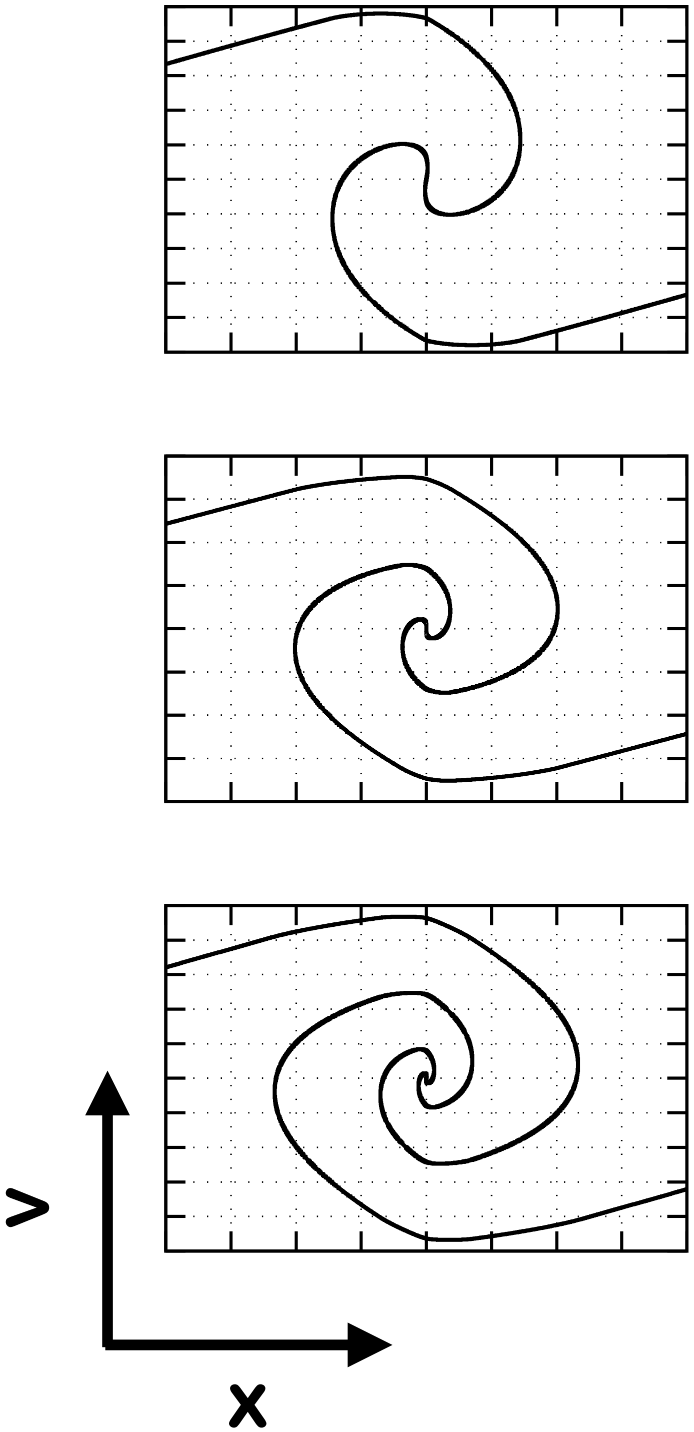
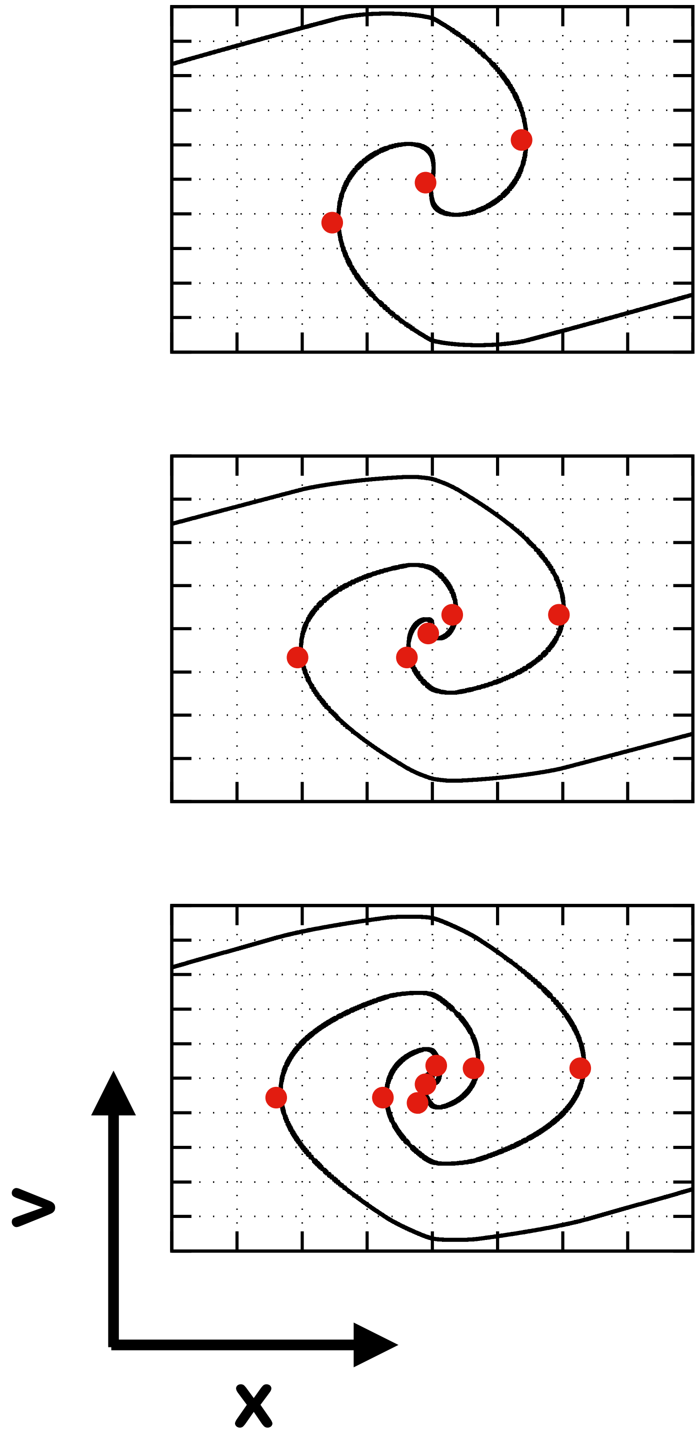
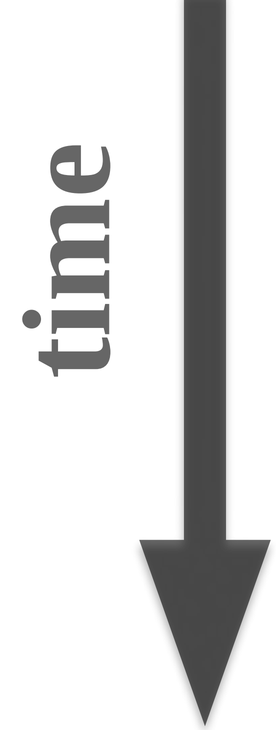

v
x





bound structures (halos) are formed

... through many infinities to come!

Key questions:

- How to theoretically model such singularities ?
- When is the first shell-crossing ?

Outline for today

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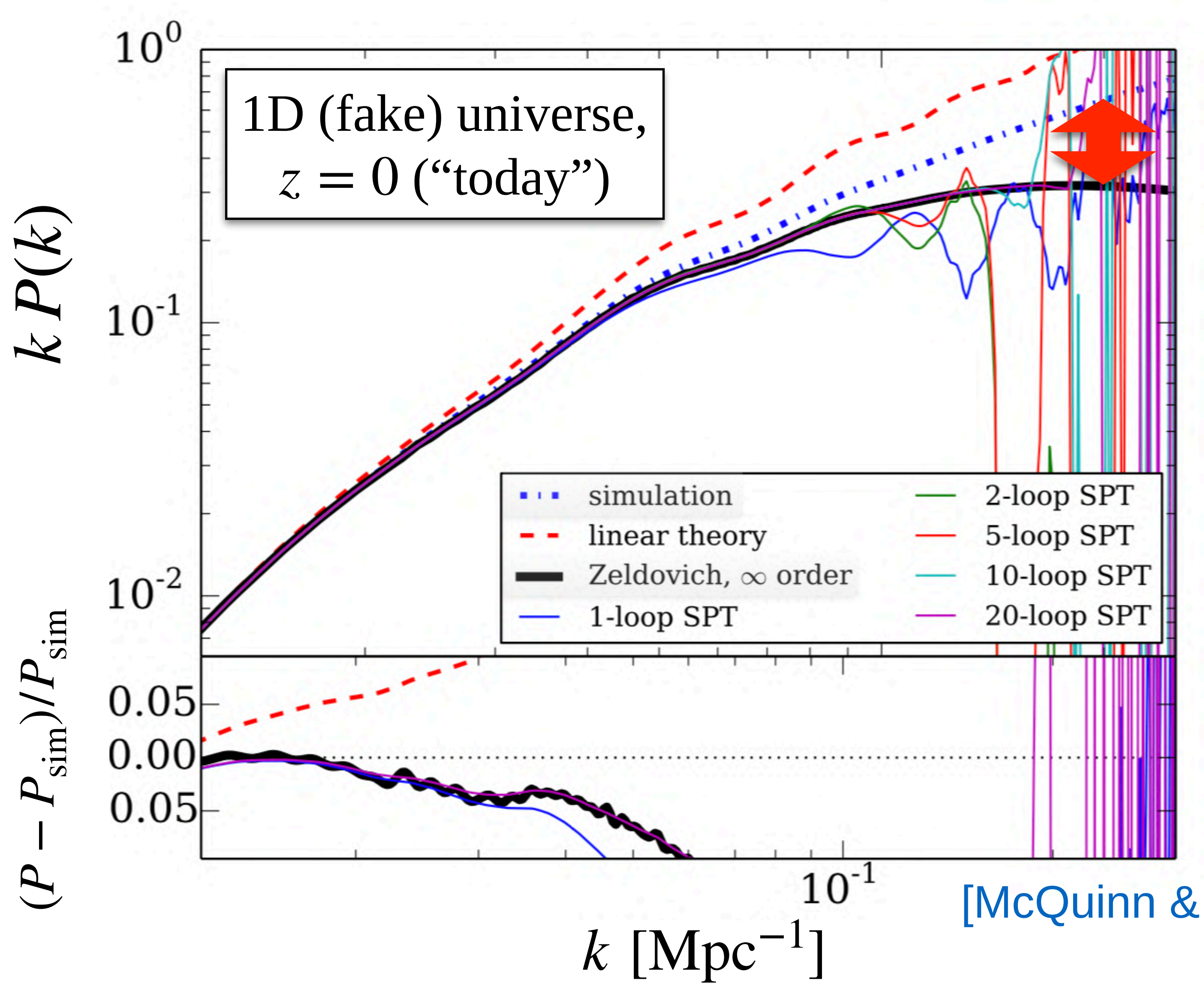
- we resolve shell-crossings:
  - first non-trivial analytical shell-crossing solutions
  - for random initial conditions, we employ very efficient semi-numerical algorithms
- recent mathematical progress makes it possible to push the modelling beyond the first shell-crossing
  - we detect so far unknown singularities in Vlasov—Poisson (associated to the known infinite densities)
  - analytical evidence is confirmed by high-resolution simulations

Shell-crossing is a key theoretical uncertainty for the matter power spectrum $P(k)$



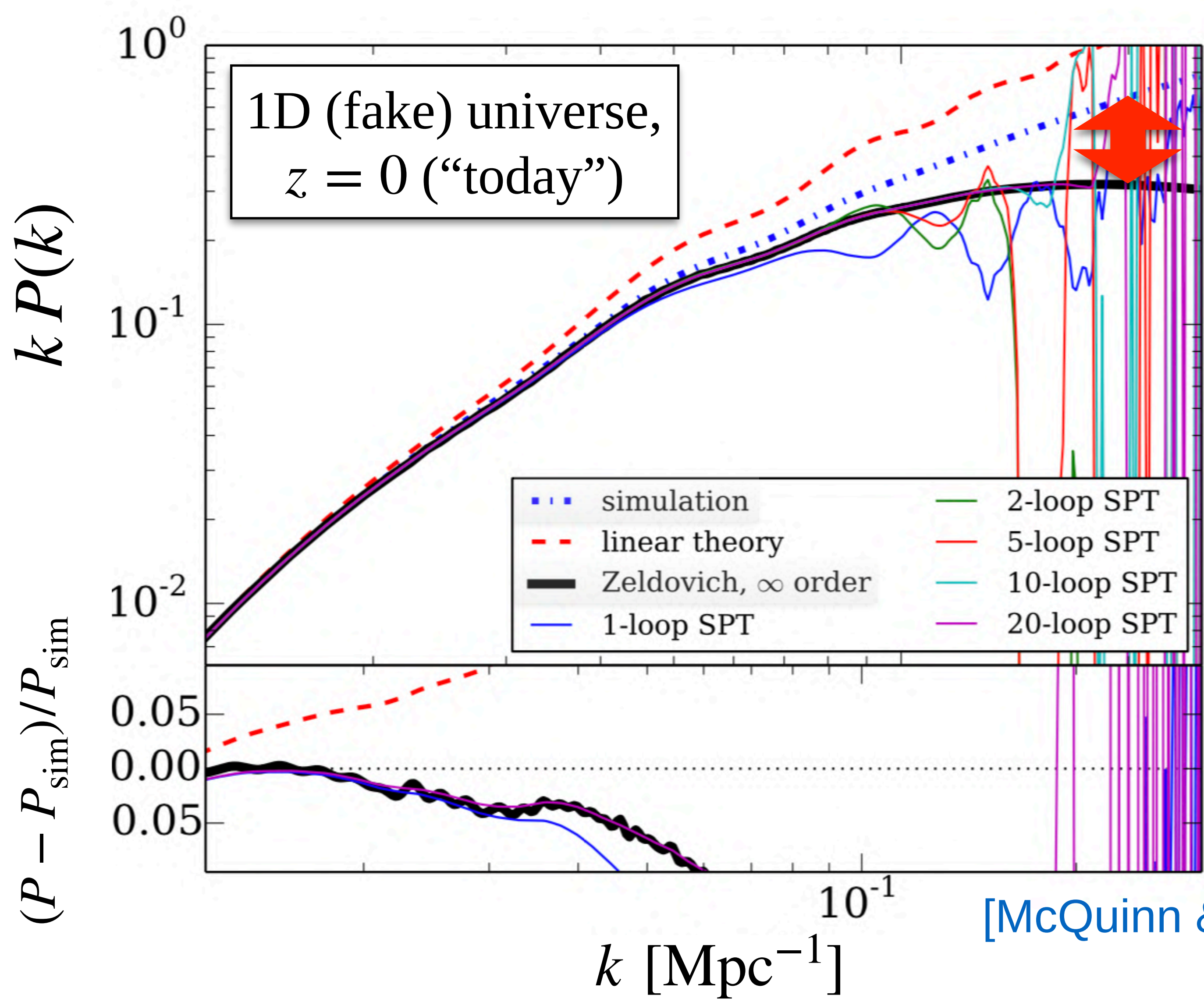
[McQuinn & White '15]

$$\left\langle \rho(\boldsymbol{k}_1)\, \rho(\boldsymbol{k}_2) \right\rangle_c \sim \delta_D(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, P(k_1)$$

In 1D, the theoretical **Zeldovich solution** is exact until shell-crossing (see later)

Motivation

Shell-crossing is a key theoretical uncertainty for the matter power spectrum $P(k)$



[McQuinn & White '15]

$$\left\langle \rho(\boldsymbol{k}_1)\, \rho(\boldsymbol{k}_2) \right\rangle_{\rm c} \sim \delta_{\rm D}(\boldsymbol{k}_1 + \boldsymbol{k}_2)\, P(k_1)$$

In 1D, the theoretical **Zeldovich solution** is exact until shell-crossing (see later)

Theoretical insight into the highly non-linear problem is relevant to

- provide accurate initial conditions for numerical simulations,
- extract information from observations, etc.

Vlasov—Poisson for CDM

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$(m = 1)$

$$\frac{\mathrm{d}f(\boldsymbol{x},\boldsymbol{p},t)}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \frac{\boldsymbol{p}}{a^2} \cdot \nabla_x f - (\nabla_x \varphi) \cdot \nabla_p f = 0$$

the "pedantic" Vlasov equation

$$\nabla_x^2 \varphi = \frac{4\pi G}{a} \left[\int f \, \mathrm{d}^3 p - \bar{\rho} \right]$$

coupled to an effective Poisson equation

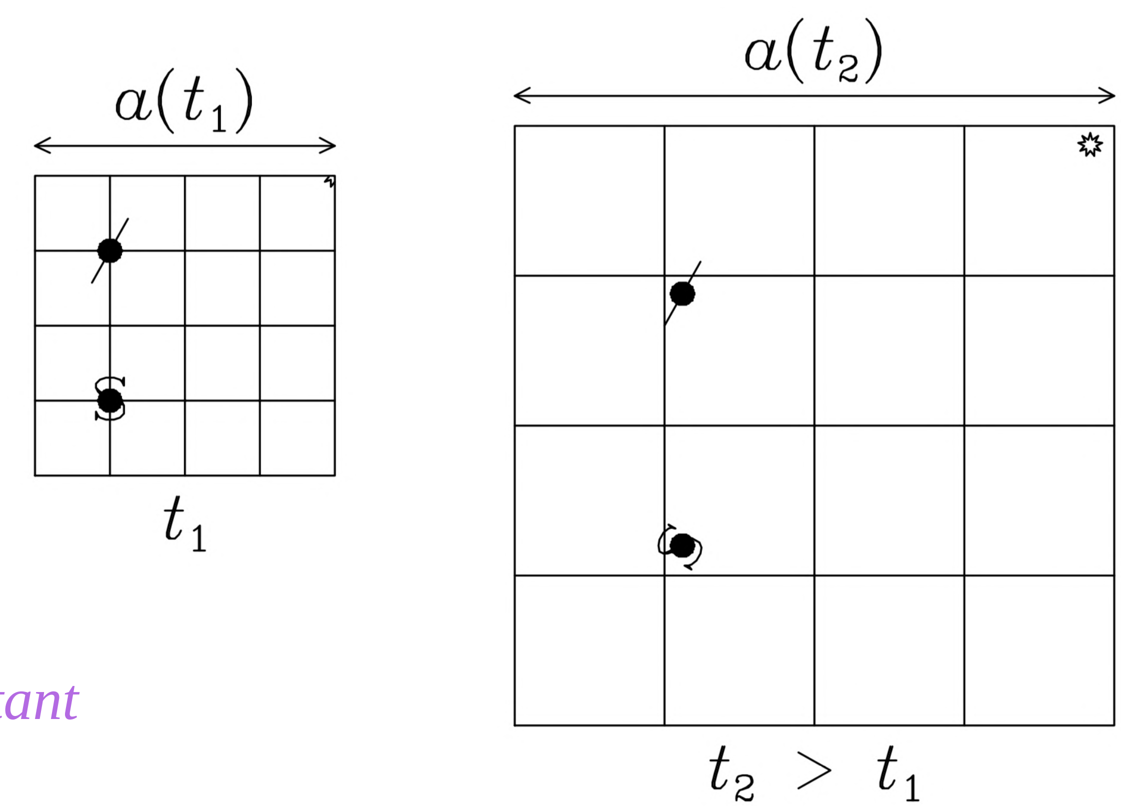
$\boldsymbol{x} = \boldsymbol{r}/a(t)$ are comoving coordinates,
$a(t)$ is cosmological scale factor determined via

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \bar{\rho}(t) + \Lambda$$

Hubble parameter

mean matter density of the Universe

cosmological constant



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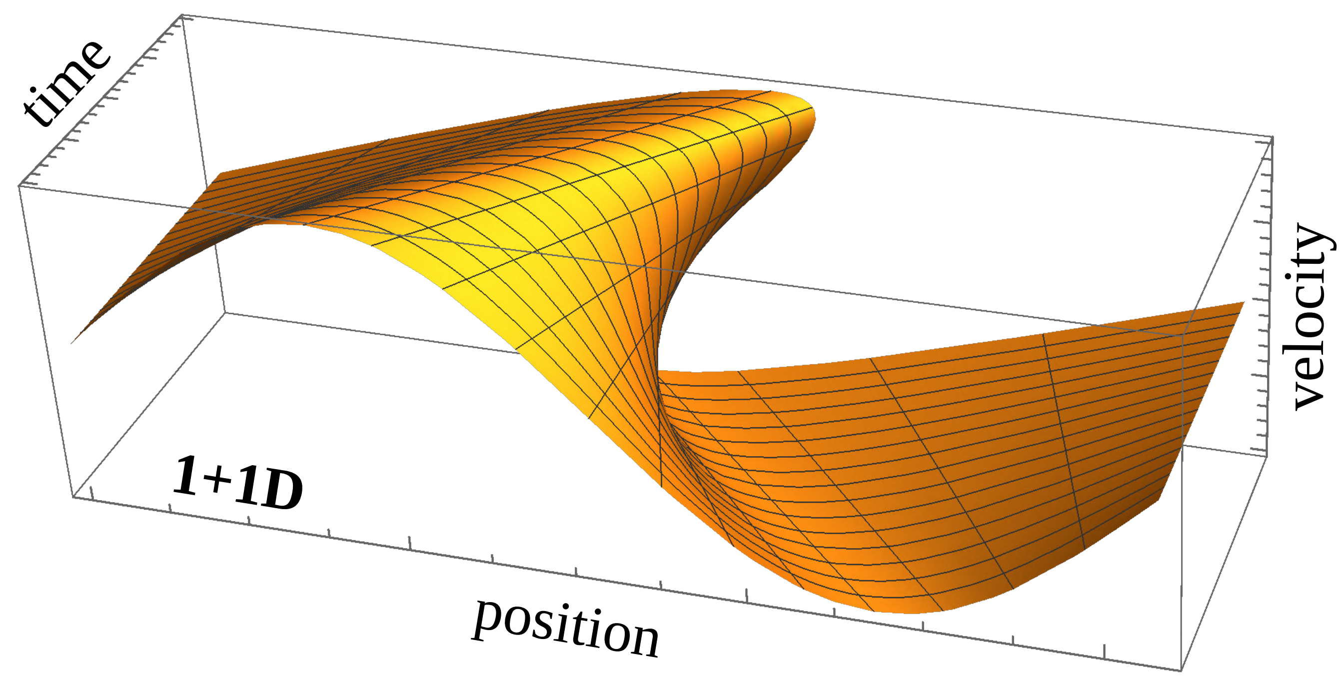
For CDM, however, $f(\boldsymbol{x},\boldsymbol{p},t)$ only occupies a **3D hypersurface in phase-space,** the Lagrangian submanifold:



Thus, a "full" 7D (space, momentum, time) description is overkill

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Parametrise CDM phase-space with the Lagrangian map $\mathbb{R}^3 \to \mathbb{R}^6 \;:\; \boldsymbol{q} \mapsto (\boldsymbol{x}, \boldsymbol{v})$

with velocity $\boldsymbol{v} = \partial_t \boldsymbol{x}$

initial position of CDM

Vlasov—Poisson (VP) for CDM reduces exactly to

$$\ddot{\boldsymbol{x}}(\boldsymbol{q},t) + 2H(t)\,\dot{\boldsymbol{x}}(\boldsymbol{q},t) = -\,\nabla_x\,\varphi(\boldsymbol{x}(\boldsymbol{q},t))$$

$$\nabla_x^2 \varphi(\boldsymbol{x}(\boldsymbol{q},t)) = \frac{4\pi G}{a}\left(\int \mathrm{d}^3q'\,\delta_{\rm D}^{(3)}(\boldsymbol{x}(\boldsymbol{q},t) - \boldsymbol{x}(\boldsymbol{q}',t)) - 1\right)$$

Solved by:

- cosmological N-body simulations using an N-particle approximation
- analytical Lagrangian-coordinates approaches in the continuum limit

N-body simulations…

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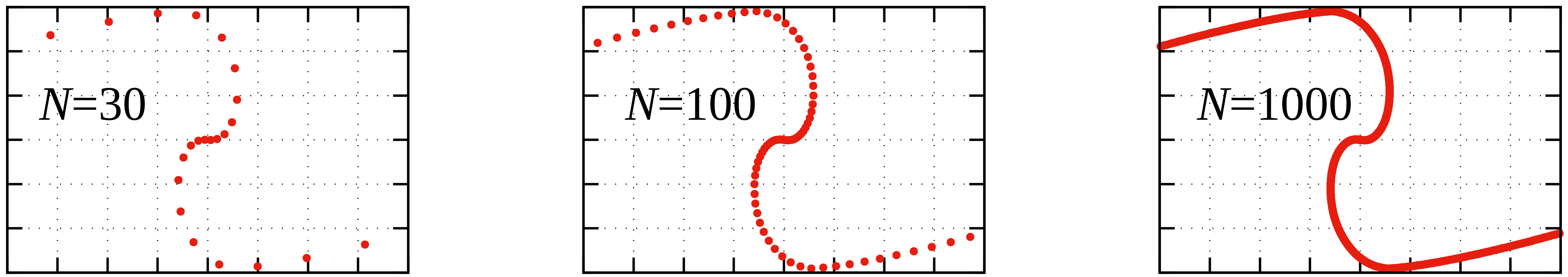
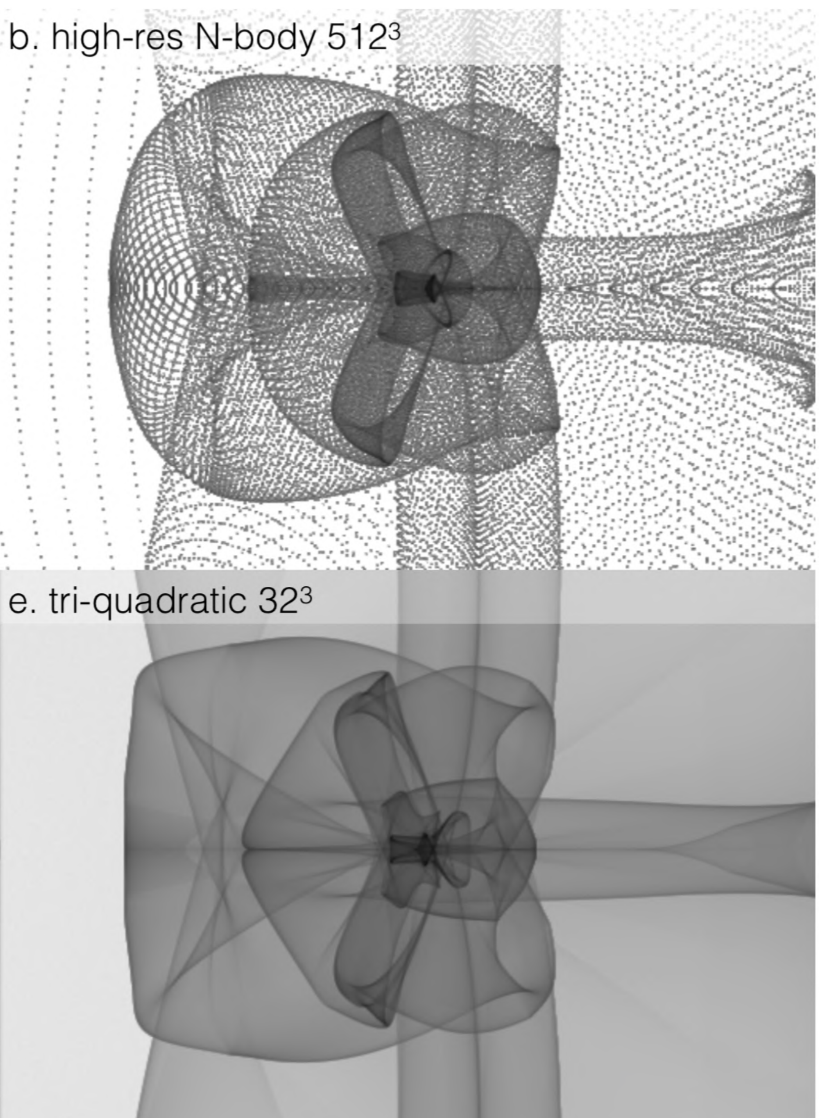
coarse-sample phase space



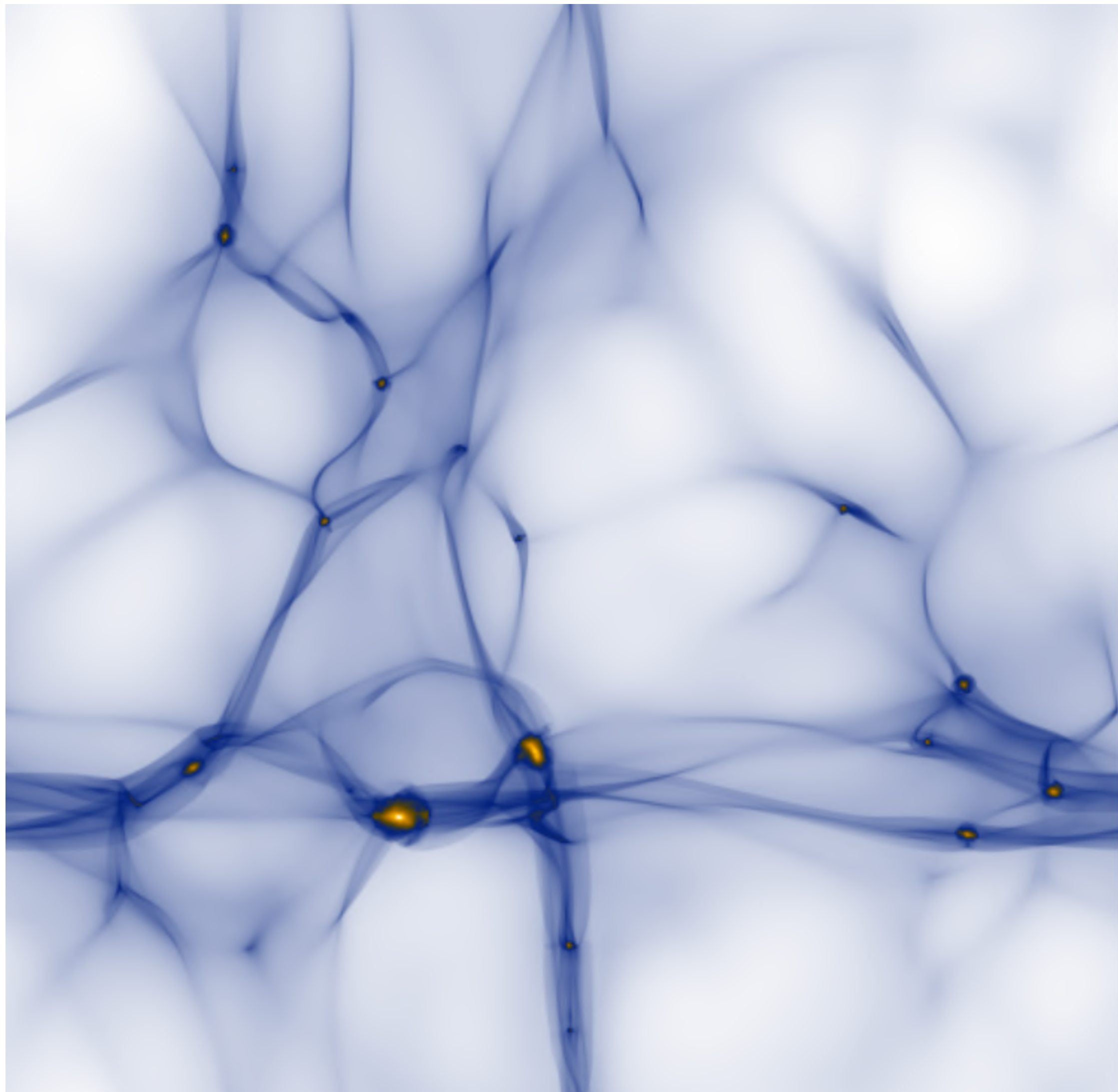
continuum limit ($N \to \infty$) may also be obtained by **tessellating** the phase-space sheet:



density snapshots

← ripple wave test problem

matter simulation →



[Hahn & Angulo 2016]

Analytical approaches

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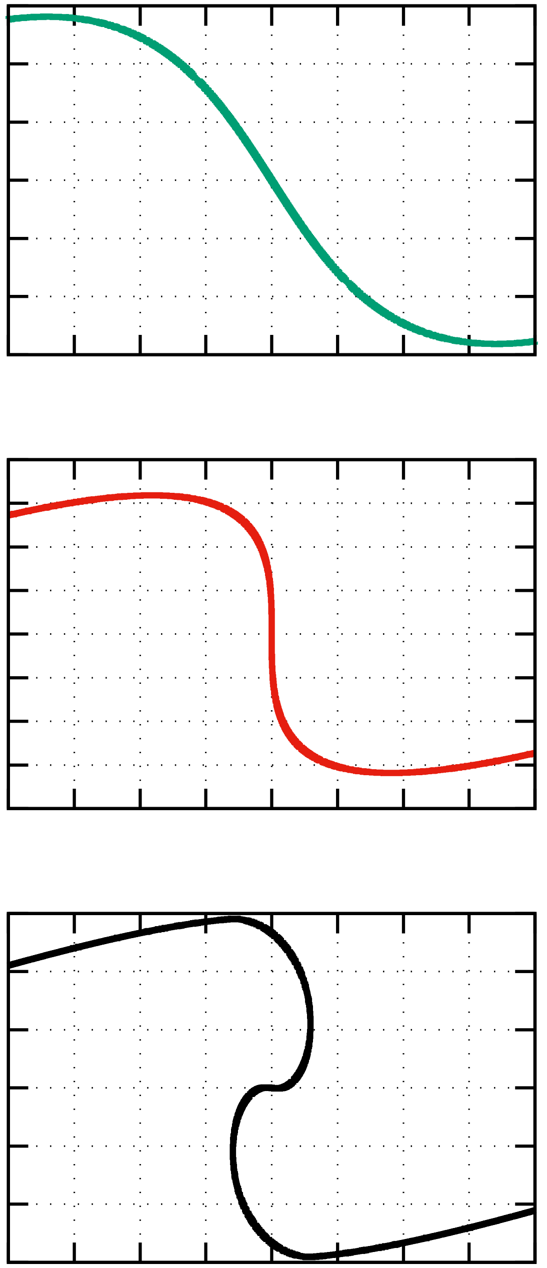
Until shell-crossing: *(see following slides)*

- VP reduces exactly to <u>cosmic fluid equations</u>
- avoid Eulerian approaches due to infinite densities
- τ time-analytic solutions with Lagrangian perturbation theory
  - *τ is dimensionless time variable* ($\propto$ *a in Einstein-de Sitter universe*)
  - *central quantity is the displacement field* $\boldsymbol{\xi}(\boldsymbol{q},\tau) := \boldsymbol{x}(\boldsymbol{q},\tau) - \boldsymbol{q}$

phase-space



BIFURCATION

After shell-crossing: *(see later)*

- fluid equations break down due to commencing bifurcation
- need <u>shell-crossing solutions</u> to provide boundary conditions
- solve Lagrangian multi-stream eqs. with refined strategy

Solutions until shell-crossing

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Continuity eq. $$\partial_\tau \delta + \nabla \cdot [1+\delta]\boldsymbol{v} = 0$$

peculiar velocity (w.r.t. background)

Euler eq. $$\partial_\tau \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} = -\frac{3}{2\tau}\left(\boldsymbol{v} + \nabla \varphi\right)$$

$\delta := (\rho - \bar{\rho})/\bar{\rho}$

Poisson eq. $$\nabla^2 \varphi = \delta/\tau$$

τ is dimensionless time variable (= a in the Einstein-de Sitter model)

Remain regular for $\tau \to 0$ iff one imposes the *slaved* boundary conditions

$$\delta^{\rm ini} = 0\,, \qquad \boldsymbol{v}^{\rm ini} = -\nabla \varphi^{\rm ini}$$

[Brenier'87; Brenier, Frisch, Hénon++ '03]

- selects growing-mode solutions, and $\nabla \times \boldsymbol{v} = 0$
- provides the foundation of power series expansion around $\tau = 0$

e.g., in Eulerian coordinates: set $\delta = \sum_{n=1}^{\infty} \delta^{(n)}(\boldsymbol{x})\,\tau^n$ and $\boldsymbol{v} = \sum_{n=1}^{\infty} \boldsymbol{v}^{(n)}(\boldsymbol{x})\,\tau^n$

$\Rightarrow$ all-order recursion relations

[Goroff, Grinstein, Rey & Wise '86]

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Introduce Lagrangian map $\boldsymbol{q} \mapsto \boldsymbol{x}(\boldsymbol{q},\tau) = \boldsymbol{q} + \boldsymbol{\xi}(\boldsymbol{q},\tau)$ of initial position $\boldsymbol{q}$ (at $\tau = 0$) to current position $\boldsymbol{x}$ at time τ, solution of characteristic equation $\boldsymbol{v} = \partial_\tau^{\mathrm{L}} \boldsymbol{x}$

seminal works by Zel'dovich '69, Buchert '89, '92, Bouchet++ 92, etc.

Lagrangian time derivative

Then the fluid equations become the "real VP":

$$(\partial_\tau^{\mathrm{L}})^2\, \boldsymbol{x}(\boldsymbol{q},\tau) + \frac{3}{2\tau}\partial_\tau^{\mathrm{L}} \boldsymbol{x}(\boldsymbol{q},\tau) = -\frac{3}{2\tau}\nabla_x \varphi(\boldsymbol{x}(\boldsymbol{q},\tau))$$

$$\nabla_{x_i} = \left(\frac{\partial q_j}{\partial x_i}\right)\nabla_{q_j}$$

$$\nabla_x^2 \varphi(\boldsymbol{x}(\boldsymbol{q},\tau)) = \delta(\boldsymbol{x}(\boldsymbol{q},\tau))/\tau$$

only before shell-crossing!

$$\delta(\boldsymbol{x}(\boldsymbol{q},\tau)) + 1 = \int \mathrm{d}^3 q'\, \delta_{\mathrm{D}}^{(3)}(\boldsymbol{x}(\boldsymbol{q},t) - \boldsymbol{x}(\boldsymbol{q}',t)) = \frac{1}{\det[\nabla_q \boldsymbol{x}(\boldsymbol{q},\tau)]}$$

The *Ansatz* $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\,\tau^n$ leads to explicit all-order recursion relations

[CR'12; Zheligovsky & Frisch '14; Matsubara '15]

"Full" solution $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\,\tau^n$ naturally represented as Helmholtz—Hodge problem:

$$\nabla_q \cdot \boldsymbol{\xi}^{(n)} = \nabla \cdot \boldsymbol{v}^{\rm ini}\,\delta_1^n + \sum_{0<s<n} \frac{s^2+(s-n)^2+(n-3)/2}{2n^2+n-3}\left(\xi_{i,j}^{(n-s)}\xi_{j,i}^{(s)} - \xi_{i,i}^{(n-s)}\xi_{j,j}^{(s)}\right) - \frac{1}{6}\sum_{s_1+s_2+s_3=n} \frac{s_1^2+s_2^2+s_3^2+(n-3)/2}{n^2+(n-3)/2}\,\varepsilon_{ikl}\varepsilon_{jmn}\xi_{i,j}^{(n_1)}\xi_{k,m}^{(n_2)}\xi_{l,n}^{(n_3)}$$

$$\nabla_q \times \boldsymbol{\xi}^{(n)} = \sum_{0<s<n} \frac{n-2s}{2n}\,\nabla\xi_k^{(s)} \times \nabla\xi_k^{(n-s)}$$

summation over repeated spatial indices assumed

How long in time can we trust this solution ?

- generally depends on precise from of initial conditions (ICs)
- for fairly generic ICs, Zheligovsky & Frisch '14 and CR, Villone & Frisch '15 obtained lower bounds on the radius of convergence of $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\,\tau^n$

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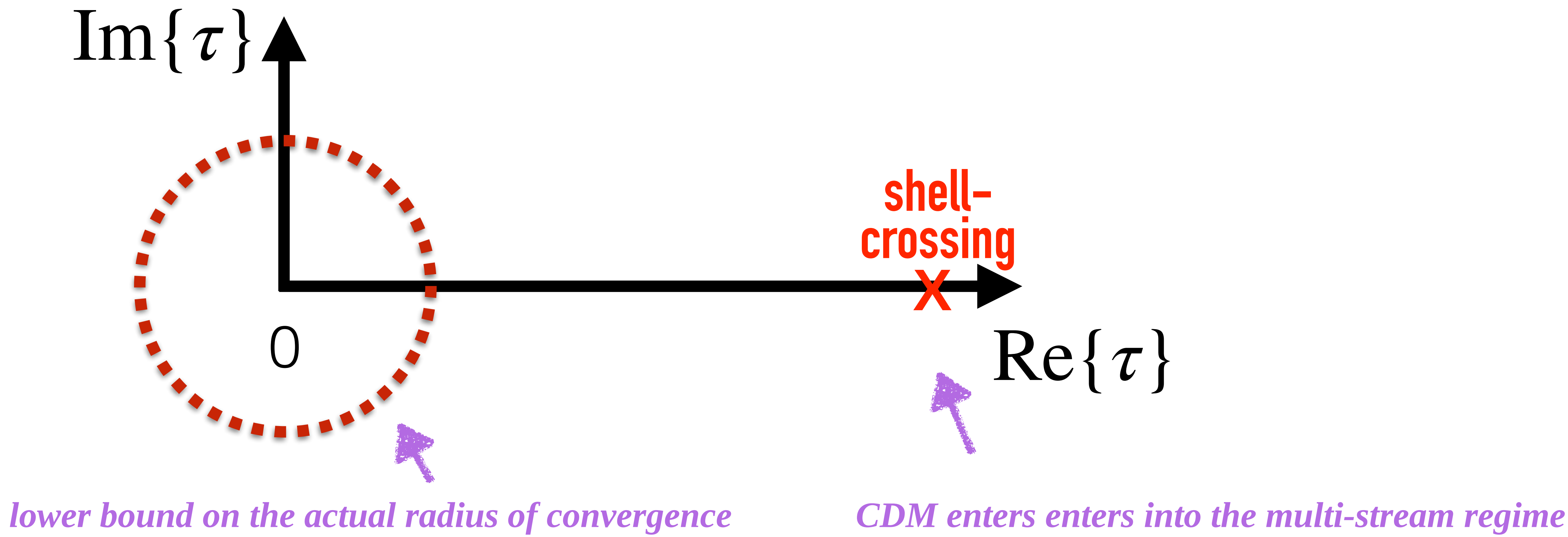
Thus, for generic ICs, the power series expansion for the displacement has a **nonzero (or infinite)** radius of convergence! The complex time domain is



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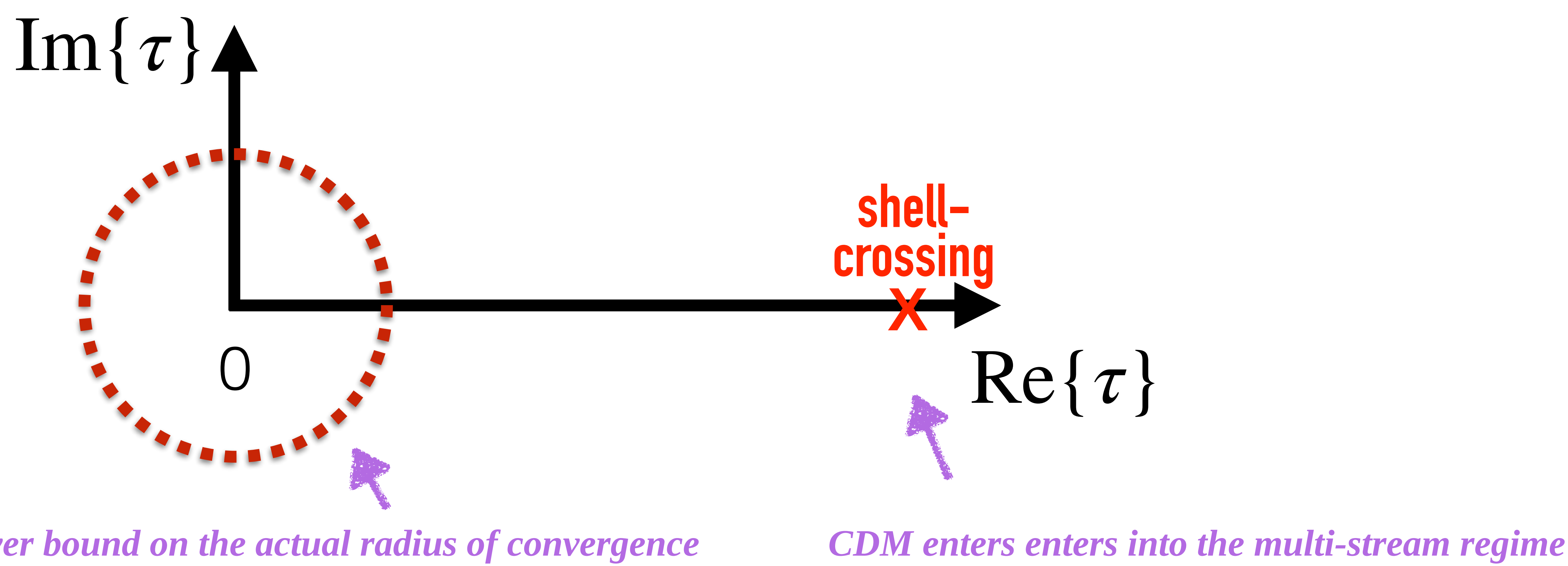


- more specific, analytical statements can be made for specialised ICs
- for generic ICs, numerical tests are required to determine the actual radius of convergence
- if shell-crossing cannot be reached in a single time step $\Rightarrow$ multi-time stepping

(seems not to be required in our case; see later)

Consider in the following the problems

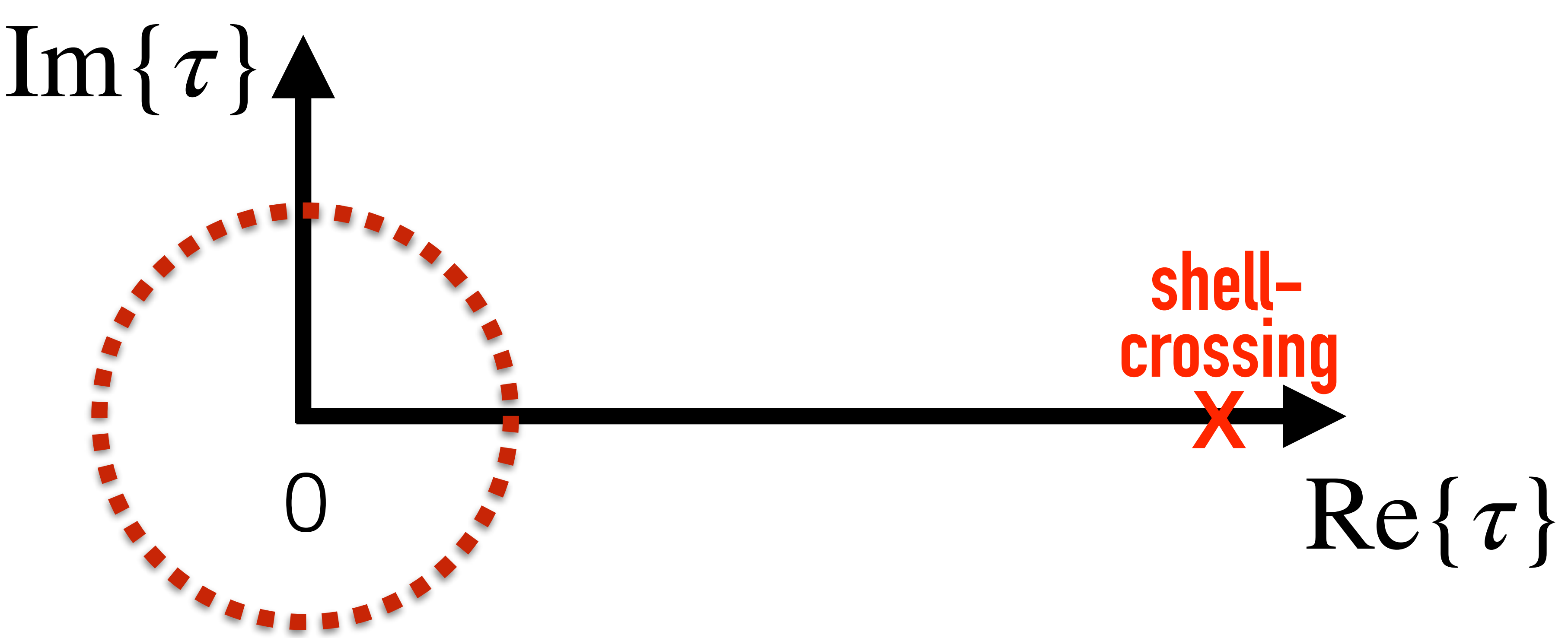
1.	one-dimensional collapse	● ○ ○ ○ ○
2.	quasi one-dimensional collapse	● ● ○ ○ ○
3.	spherical & quasi-spherical collapse	● ● ● ○ ○
4.	three-dimensional sine-wave collapse	● ● ● ● ○
5.	collapse for cosmological ICs	● ● ● ● ●

In 1D, $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \boldsymbol{\xi}^{(1)}(\boldsymbol{q})\,\tau$ is the **exact ("Zel'dovich")** solution until shell-crossing

[Novikov '69, Zel'dovich '69]

"Proof:" in Lagrangian coordinates the fluid equations are exactly linear in 1D.

Thus the linear solution is exact (map is an entire function in τ), and the mathematical radius of convergence is infinite. ■



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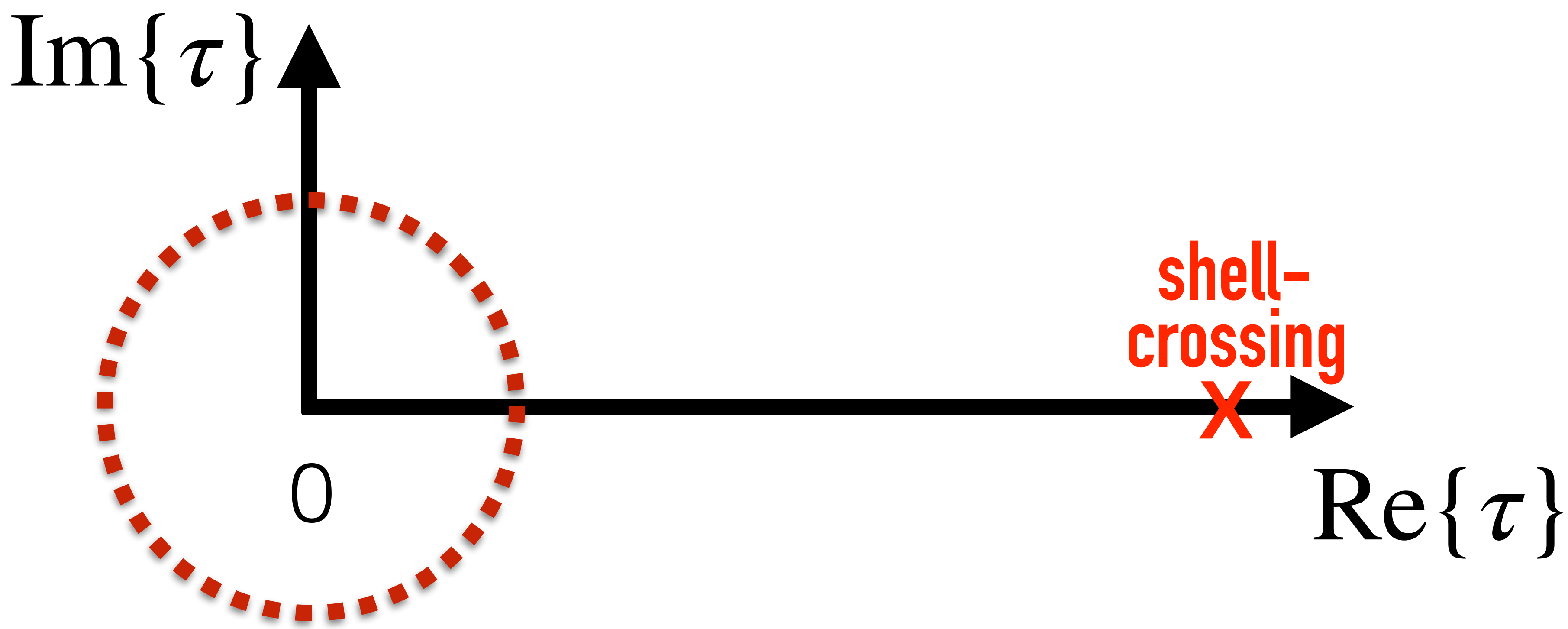

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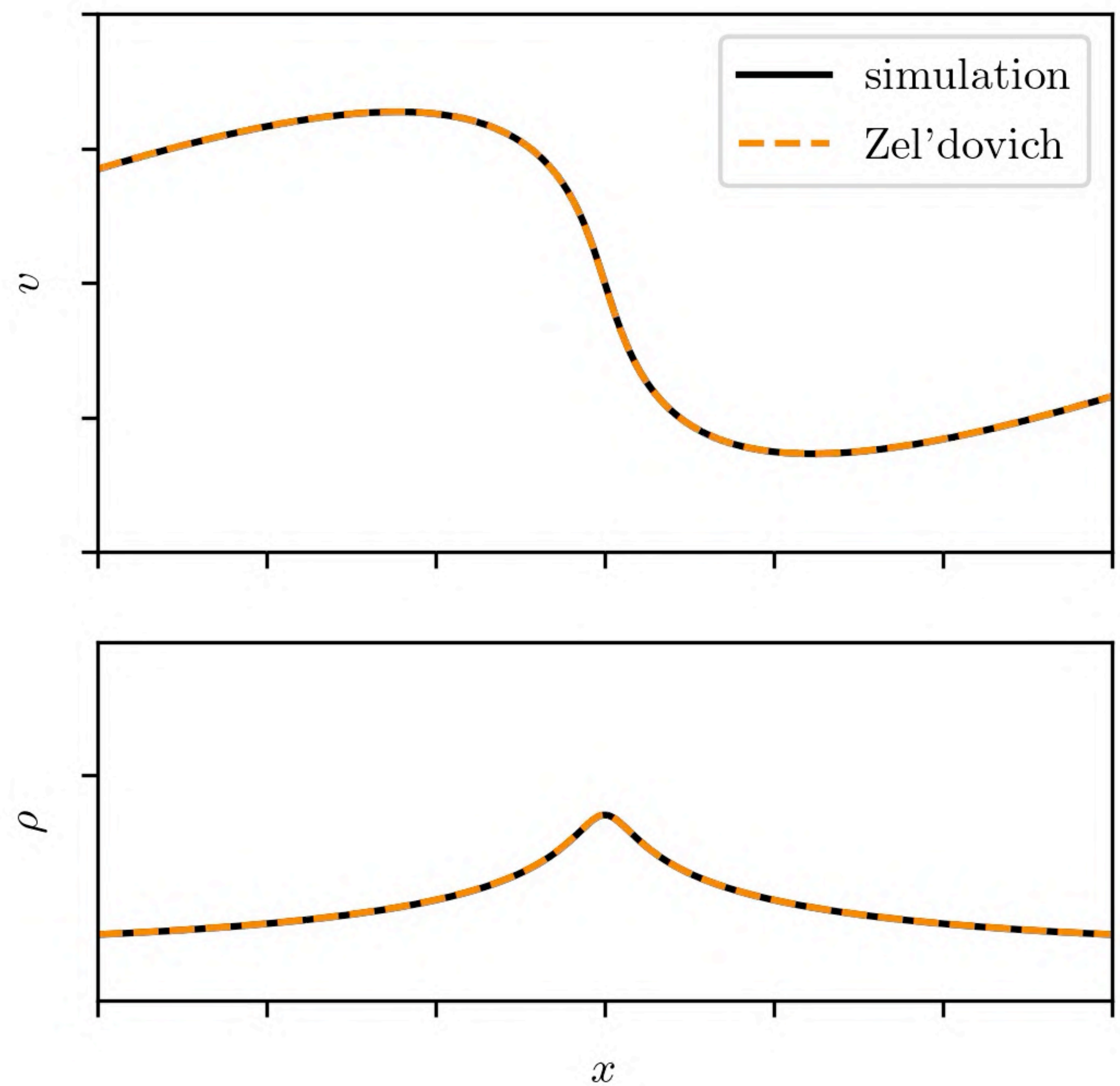
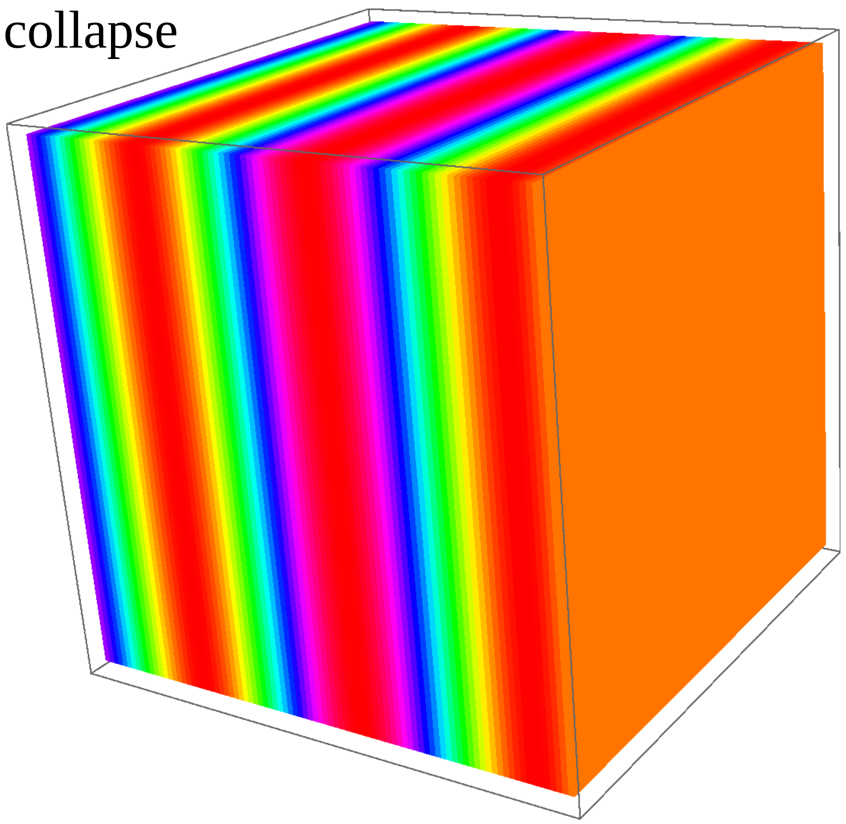
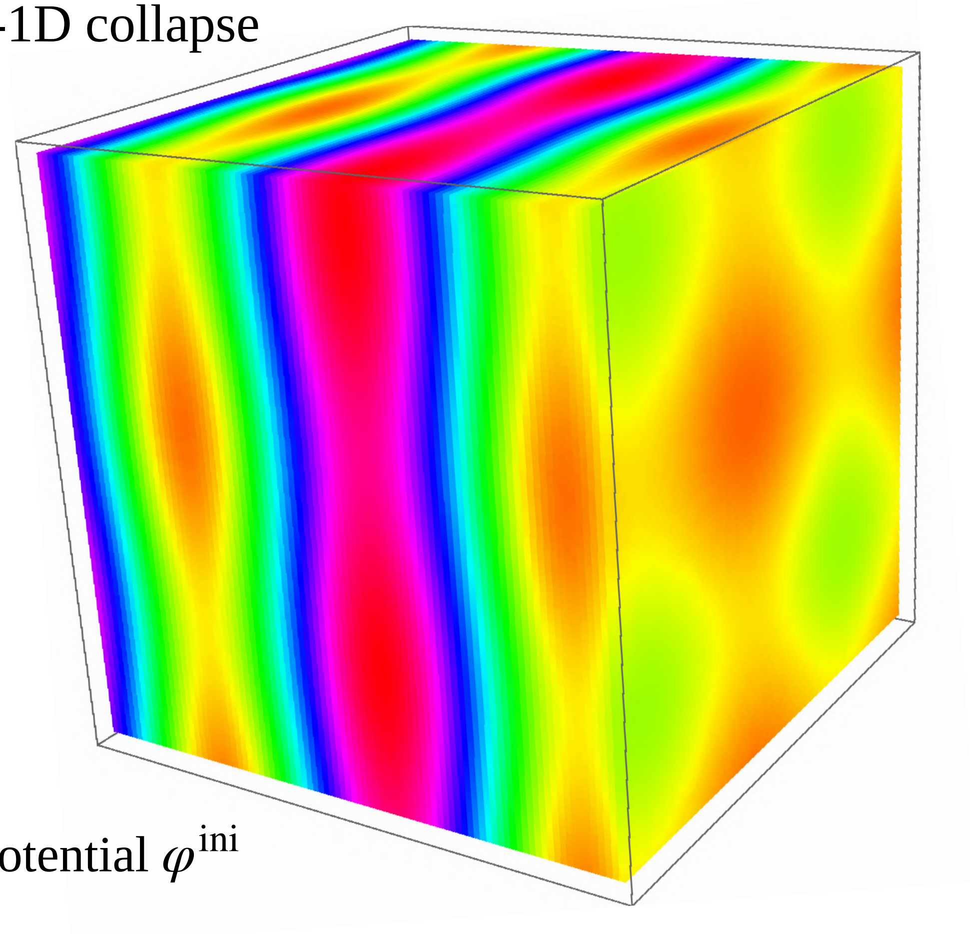
[Novikov ’69, Zel’dovich ’69]



credit: M. Bühlmann

briefly, embedding 1D and quasi 1D problems in 3D:





color hue denotes initial gravitational potential φ^{ini}

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★ Departure from 1D leads to a population of all coefficients in $\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{\infty} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\, \tau^n$, however with **very strong decay** in amplitude for $n \to \infty$

★ Radius of convergence is still infinite

★ Mathematical proof elementary, requires usage of multiscaling techniques

[CR & Frisch '17]

- ★ Spherical top-hat over-density
  Popular model in cosmology [Peebles '67]
- ★ Lagrangian power series converges slowly
  low-order solutions: [Munshi, Sahni & Starobinsky '94]

★ Spherical top-hat over-density
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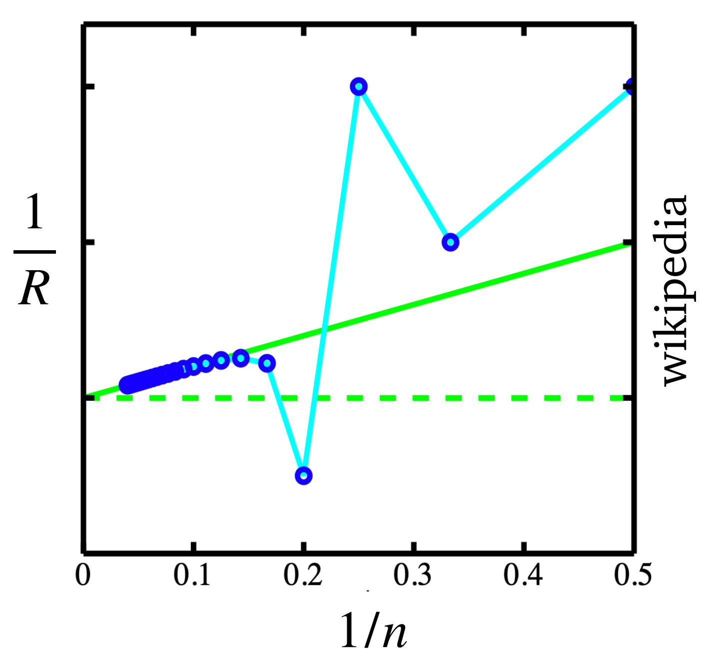
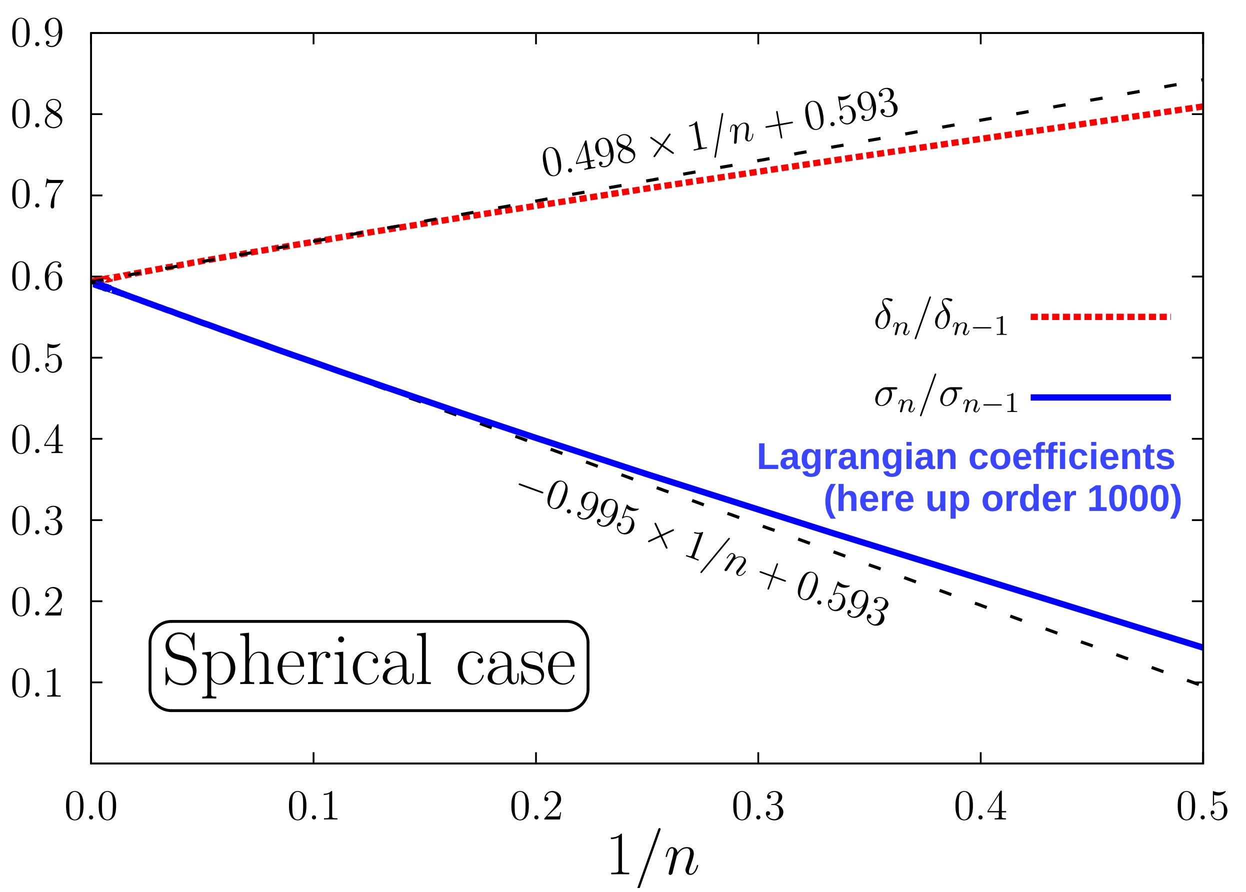
★ Lagrangian power series converges slowly
low-order solutions: [Munshi, Sahni & Starobinsky '94]

★ ratio test (Domb—Sykes) points to a singularity at shell-crossing: [CR'19]

[Domb & Sykes '57]



$$\frac{1}{R} = \lim_{n\to\infty} \frac{\sigma_n}{\sigma_{n-1}}$$



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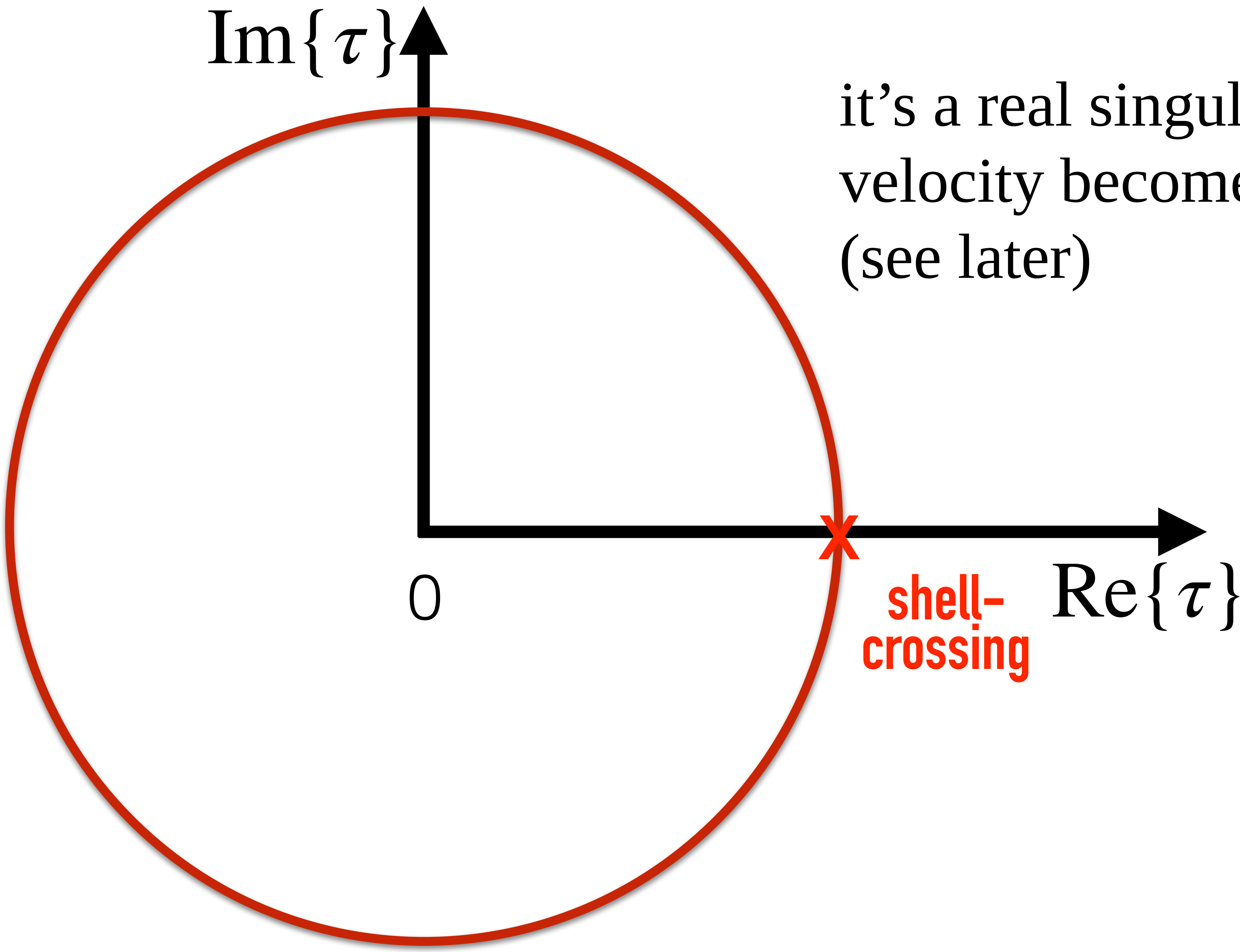
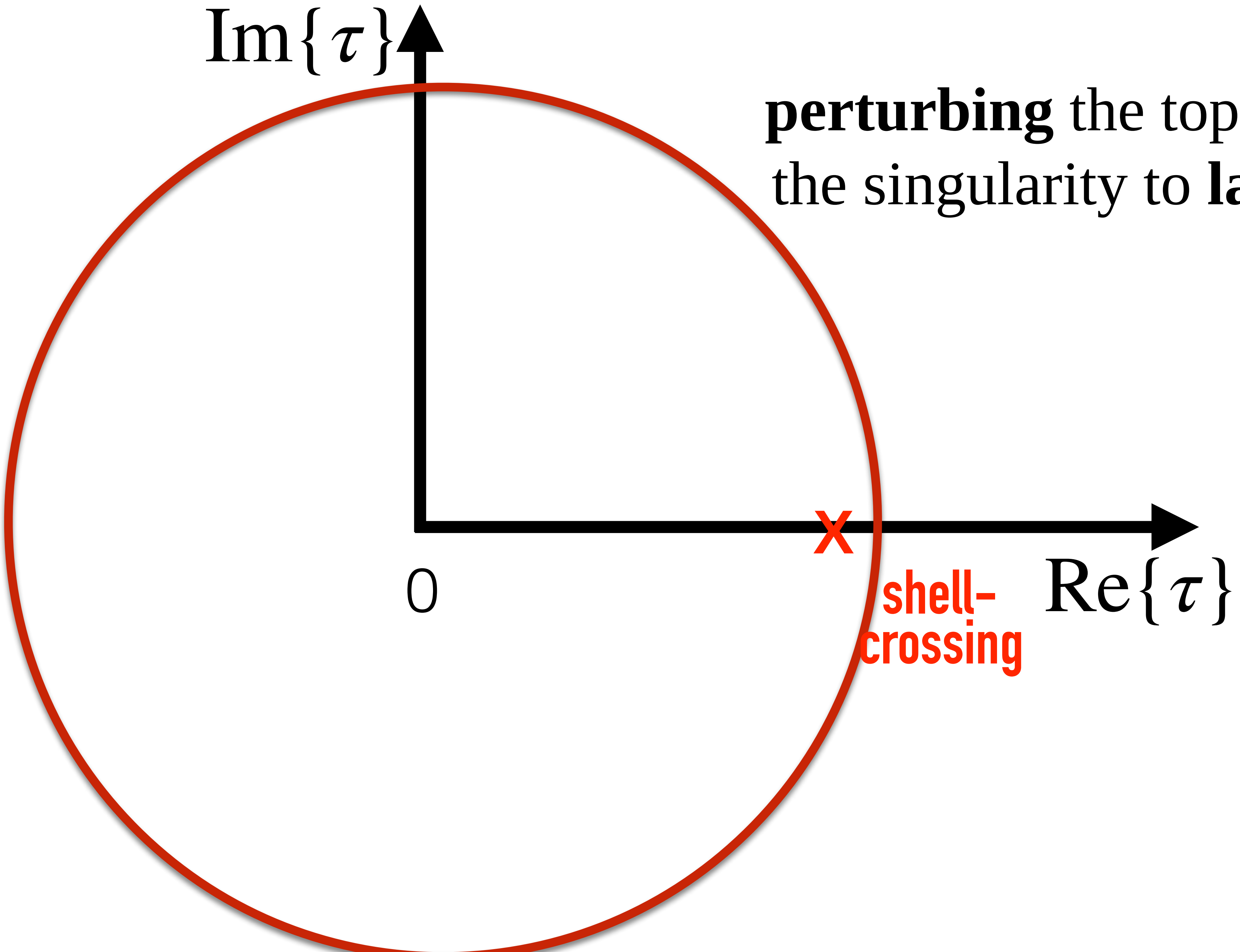


it's a real singularity:
velocity becomes infinite
(see later)

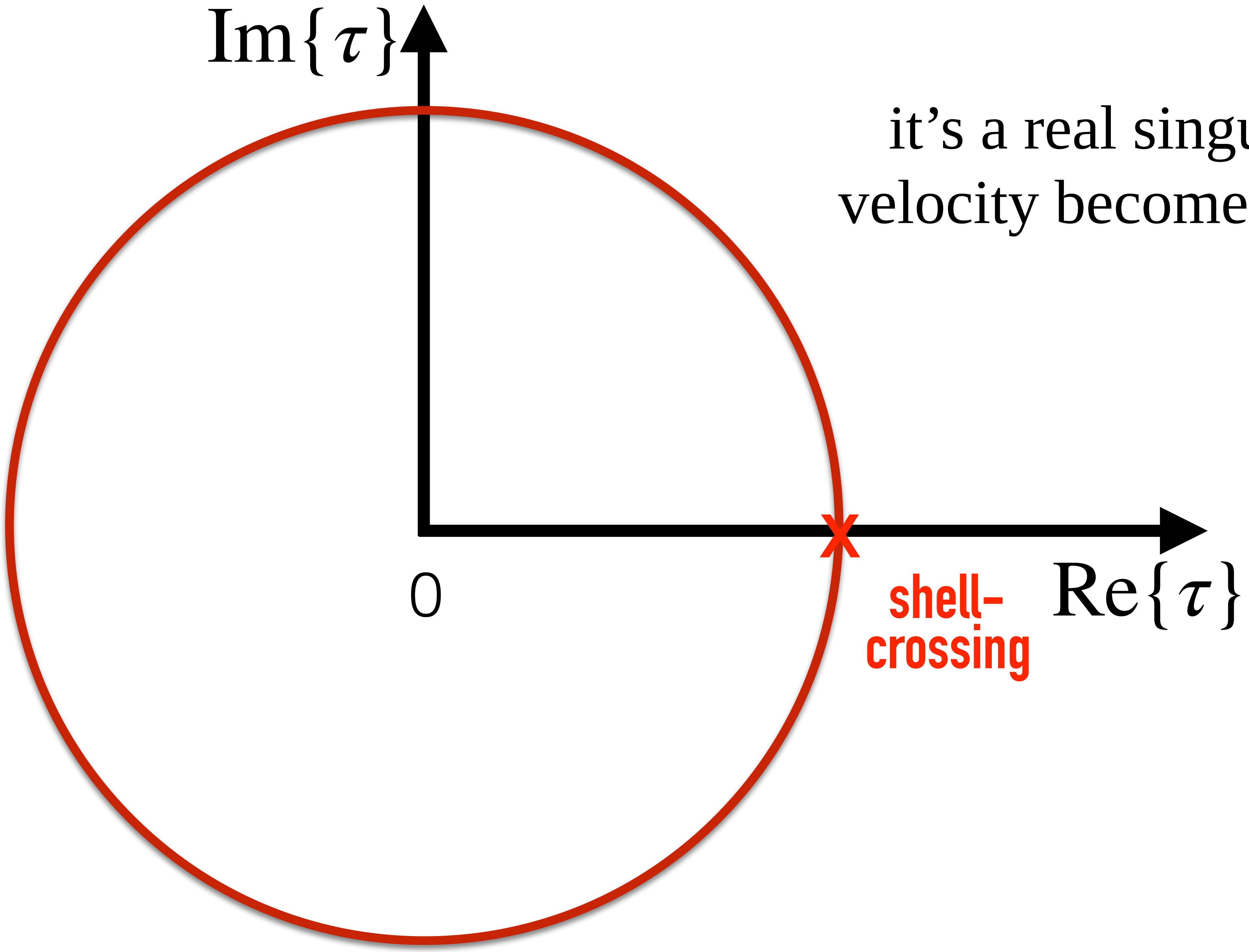
quasi-spherical top-hat

[CR '19]



perturbing the top hat **shifts** the singularity to **later** times

[CR '19]



it's a real singularity:
velocity becomes infinite

Numerical verification (sine-wave ICs)

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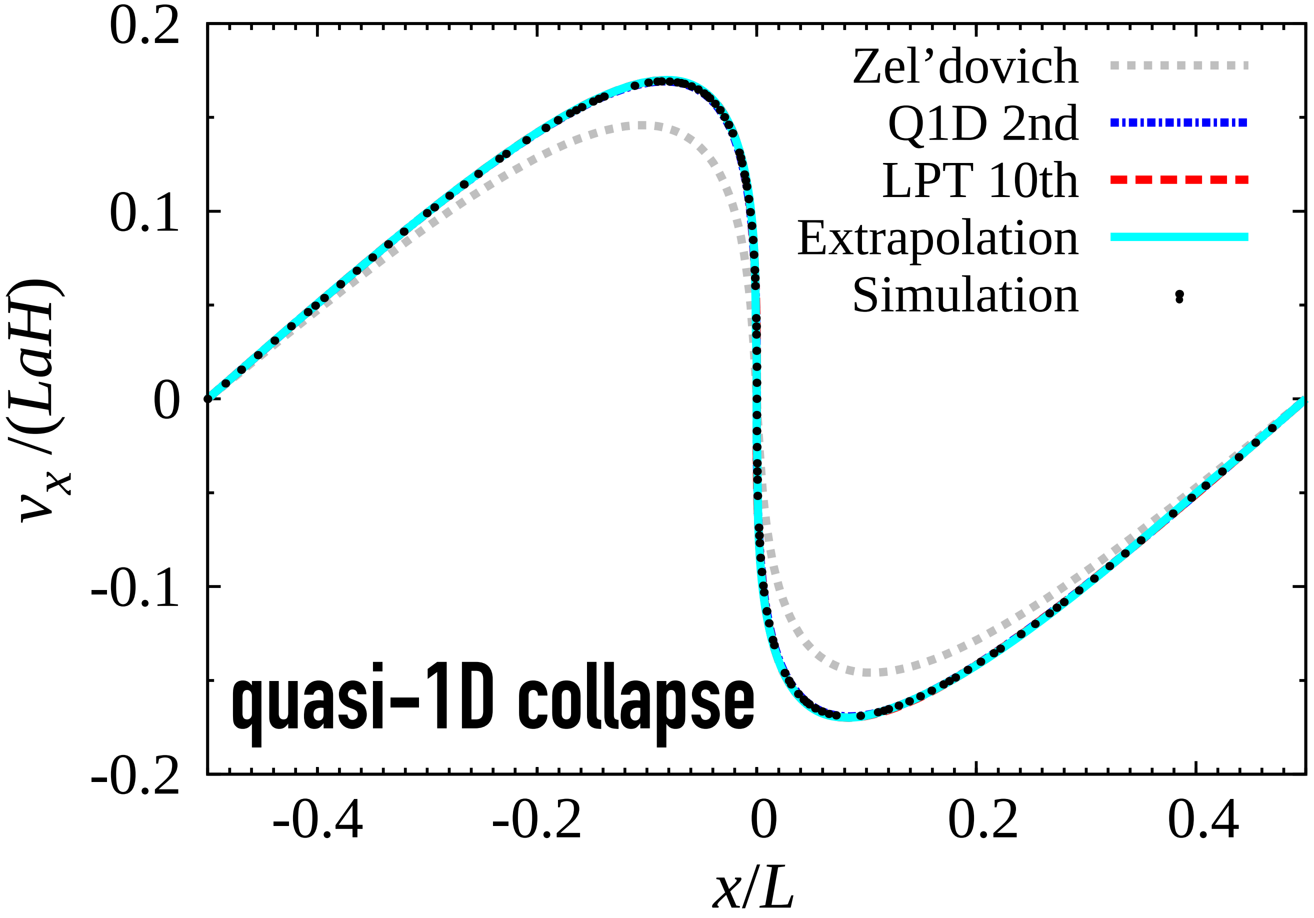
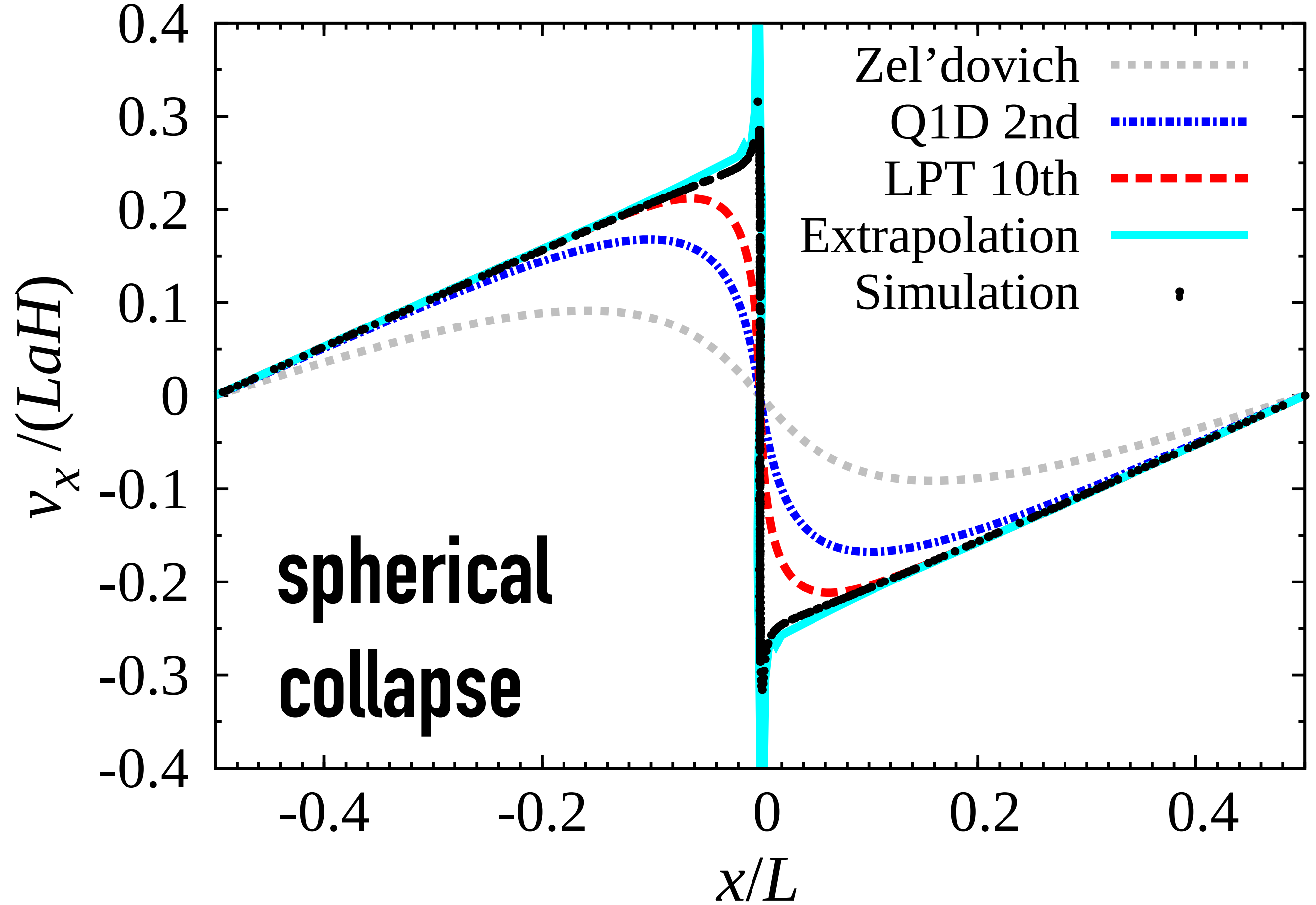
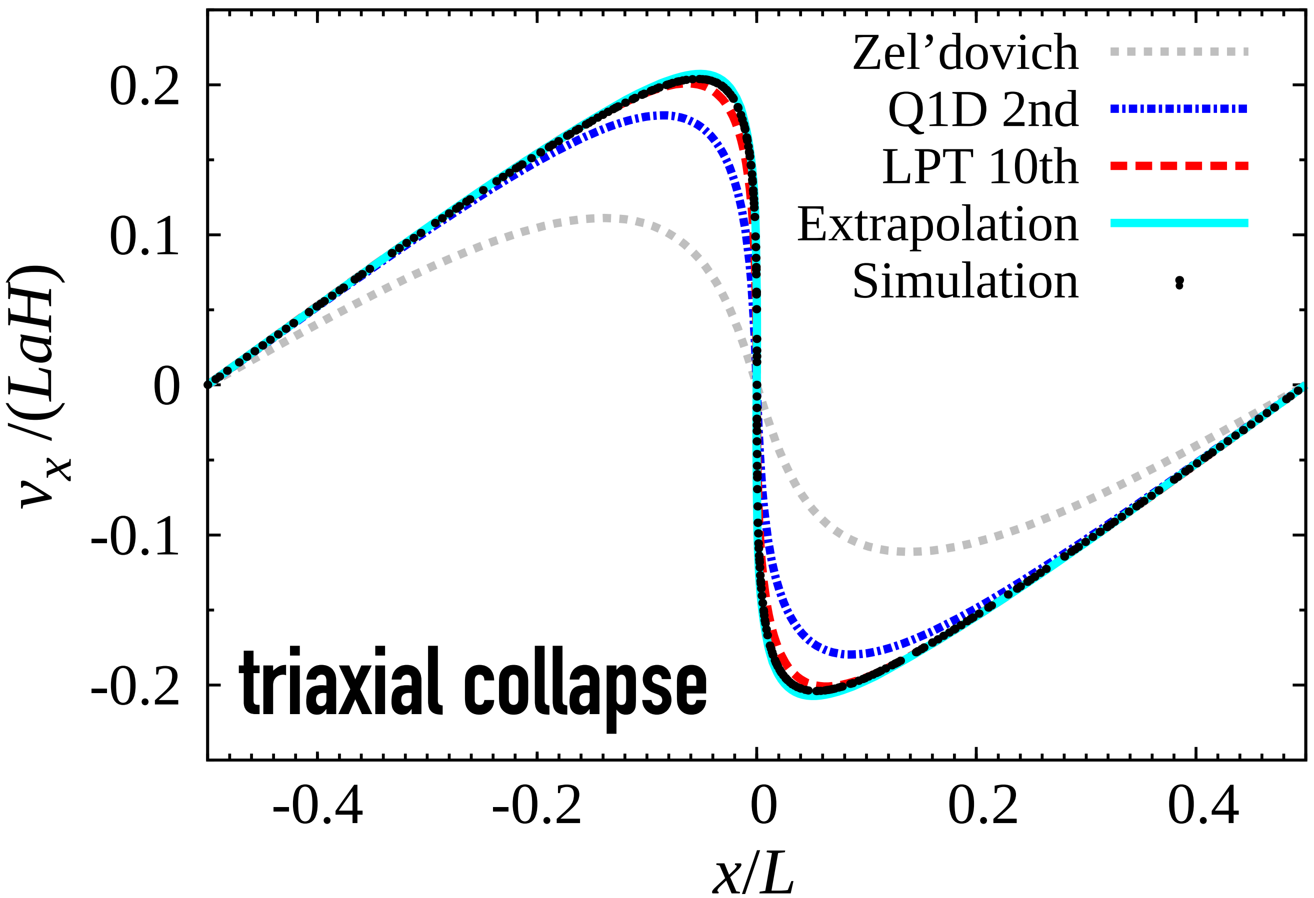
[Saga, Taruya & Colombi, PRL 2018]

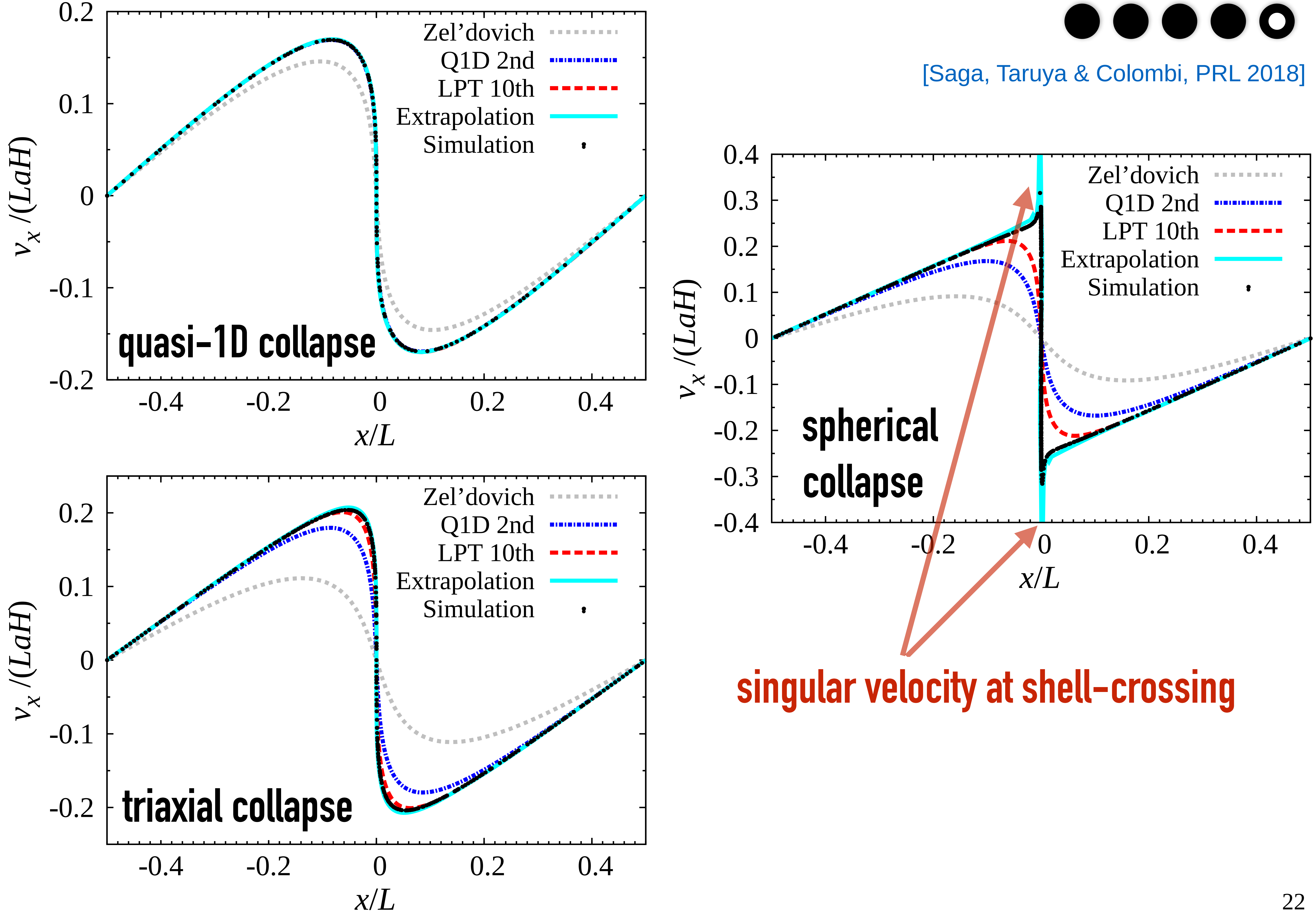






Numerical verification (sine-wave ICs)

[Saga, Taruya & Colombi, PRL 2018]



singular velocity at shell-crossing

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The standard for generating ICs for cosmological simulations is using

$$\boldsymbol{\xi}(\boldsymbol{q},\tau) = \sum_{n=1}^{n_{\rm max}} \boldsymbol{\xi}^{(n)}(\boldsymbol{q})\,\tau^n \text{ with } n_{\rm max} = 2, \textbf{ but we need at least } n_{\rm max} = 3$$

for performing the "simplest" convergence test:

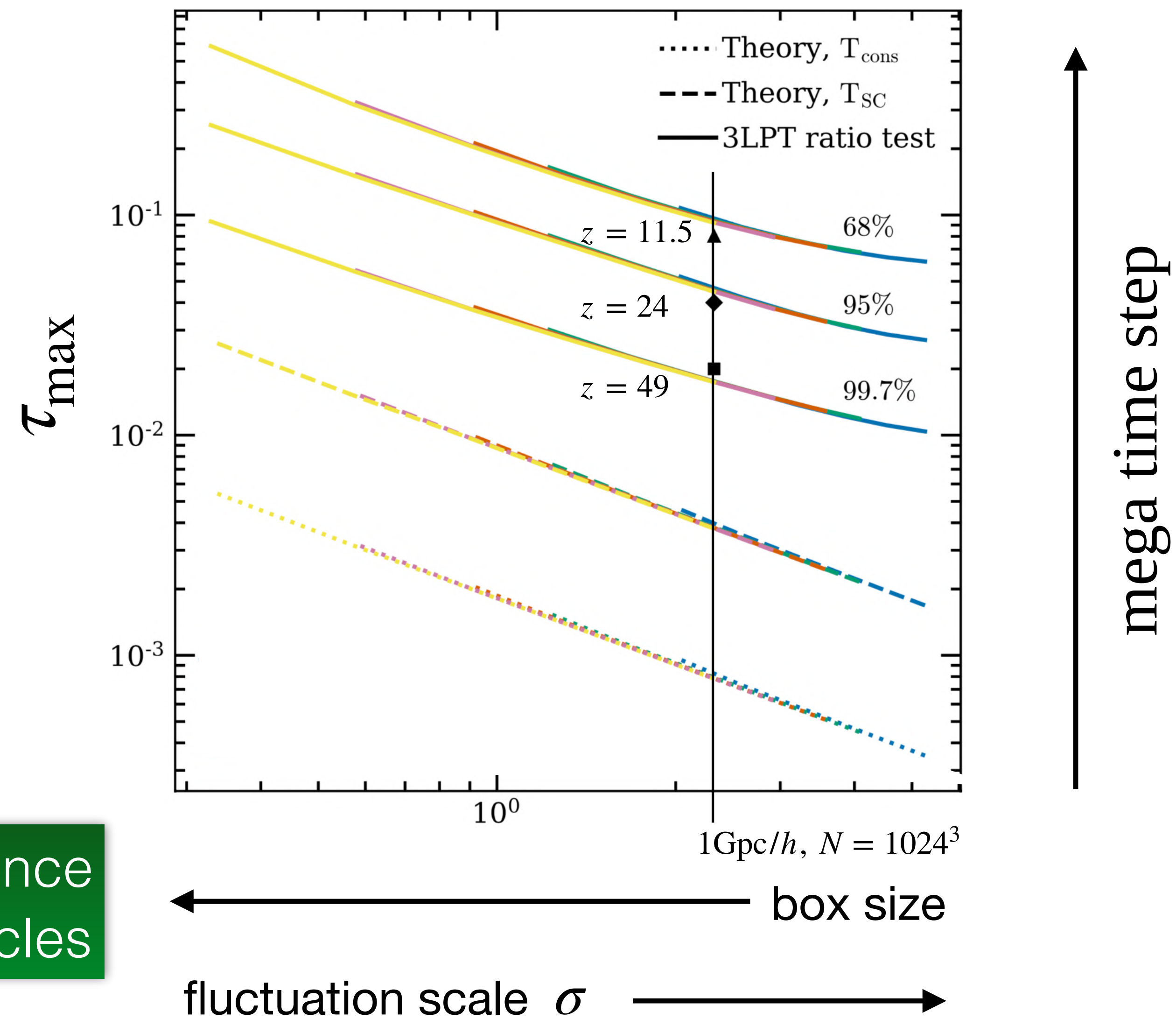
[Michaux, Hahn, CR & Angulo '20]

ratio test:

$$\frac{1}{R} = \lim_{n\to\infty} \frac{\|\boldsymbol{\xi}^{(n)}\|}{\|\boldsymbol{\xi}^{(n-1)}\|}$$

we calculate $\boldsymbol{\xi}^{(s)}$ for $s = 1,2,3$

$$\frac{1}{R} \simeq 3\frac{\|\boldsymbol{\xi}^{(3)}\|}{\|\boldsymbol{\xi}^{(2)}\|} - 2\frac{\|\boldsymbol{\xi}^{(2)}\|}{\|\boldsymbol{\xi}^{(1)}\|}$$



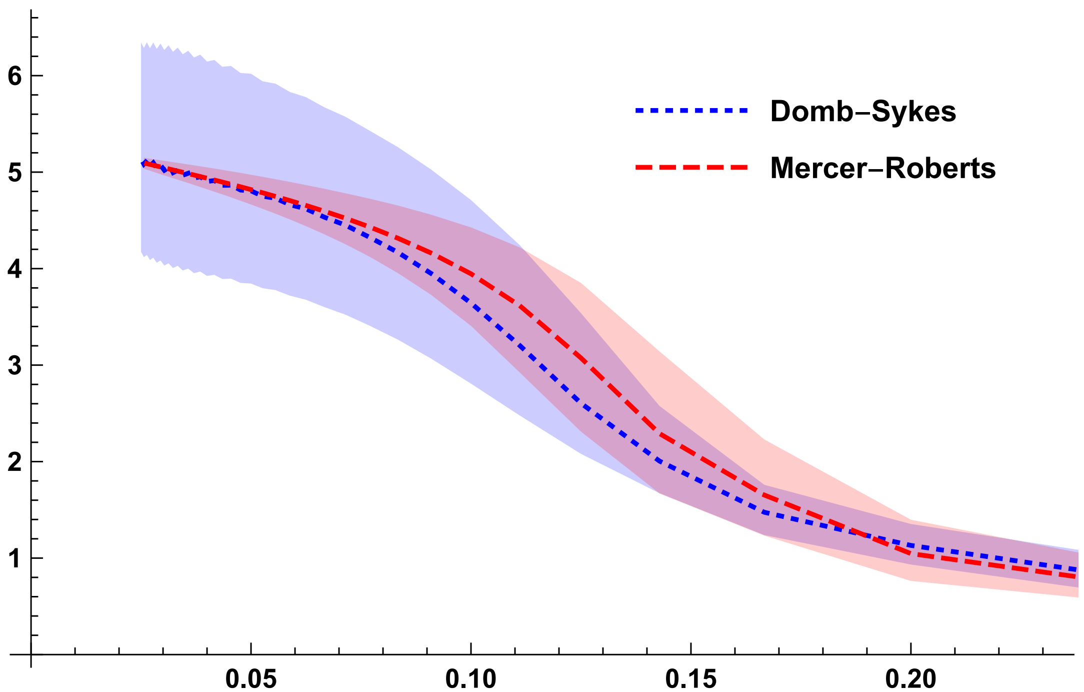
$\tau_{\rm max}$ is the latest possible time for which convergence is still guaranteed for 68%, 95%, or 99.7% of particles

[CR & Hahn, in preparation]

Numerical implementation of the Lagrangian recursion relations in `C++`

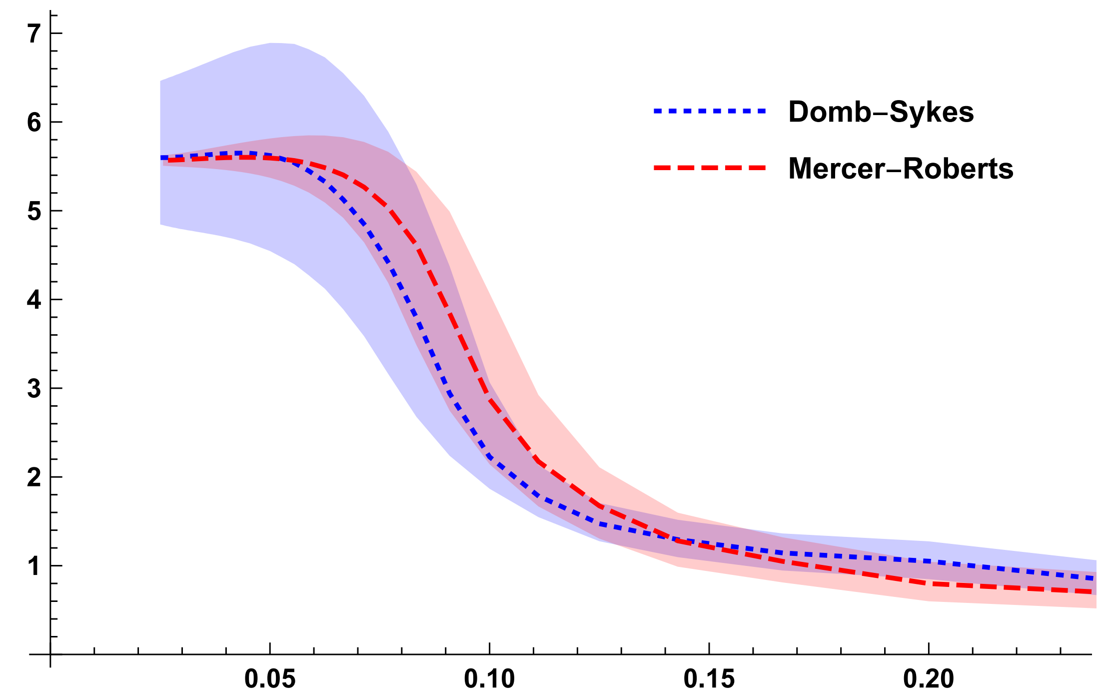
- fully de-aliased code (memory intensive, involving triple convolutions)
- supports `long double` precision, in our case `double` is sufficient for $n_{\rm max} = 40$

$N = 128^3,\ L_{\rm box} = 125\,{\rm Mpc}/h$



$1/n$

$N = 256^3,\ L_{\rm box} = 125\,{\rm Mpc}/h$



$1/n$

work in progress — but it appears that the first singularity is **after** shell-crossing

Final few slides:

beyond shell-crossing

Theories beyond shell-crossing

Cornelius Rampf
Lab. Lagrange, Nice

[Taruya & Colombi'17; Pietroni'18; CR,Frisch & Hahn'19]

$$\underbrace{\ddot{\boldsymbol{x}}(\boldsymbol{q},\tau)}_{\text{acceleration}} \propto \underbrace{-\nabla_x \varphi(\boldsymbol{x}(\boldsymbol{q},\tau))}_{\text{gravitational force}}$$

$$\nabla_x^2 \varphi \propto \sum_{n\,\text{roots}} \frac{1}{|\det x_{i,j}(\boldsymbol{q}_n)|} - 1$$

nontrivial with multi-streaming where RHS $\to \infty$;
but $\nabla_x \varphi$ remains finite in Lagrangian coordinates!

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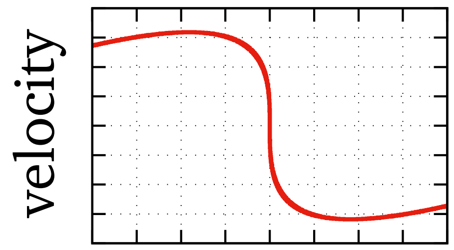
General strategy:

1. **solve** for the trajectories $\boldsymbol{x}_{\rm sc}$ until the first shell-crossing with LPT
2. provide boundary conditions **at shell-crossing**
3. solve multi-stream equations with **refined** strategy

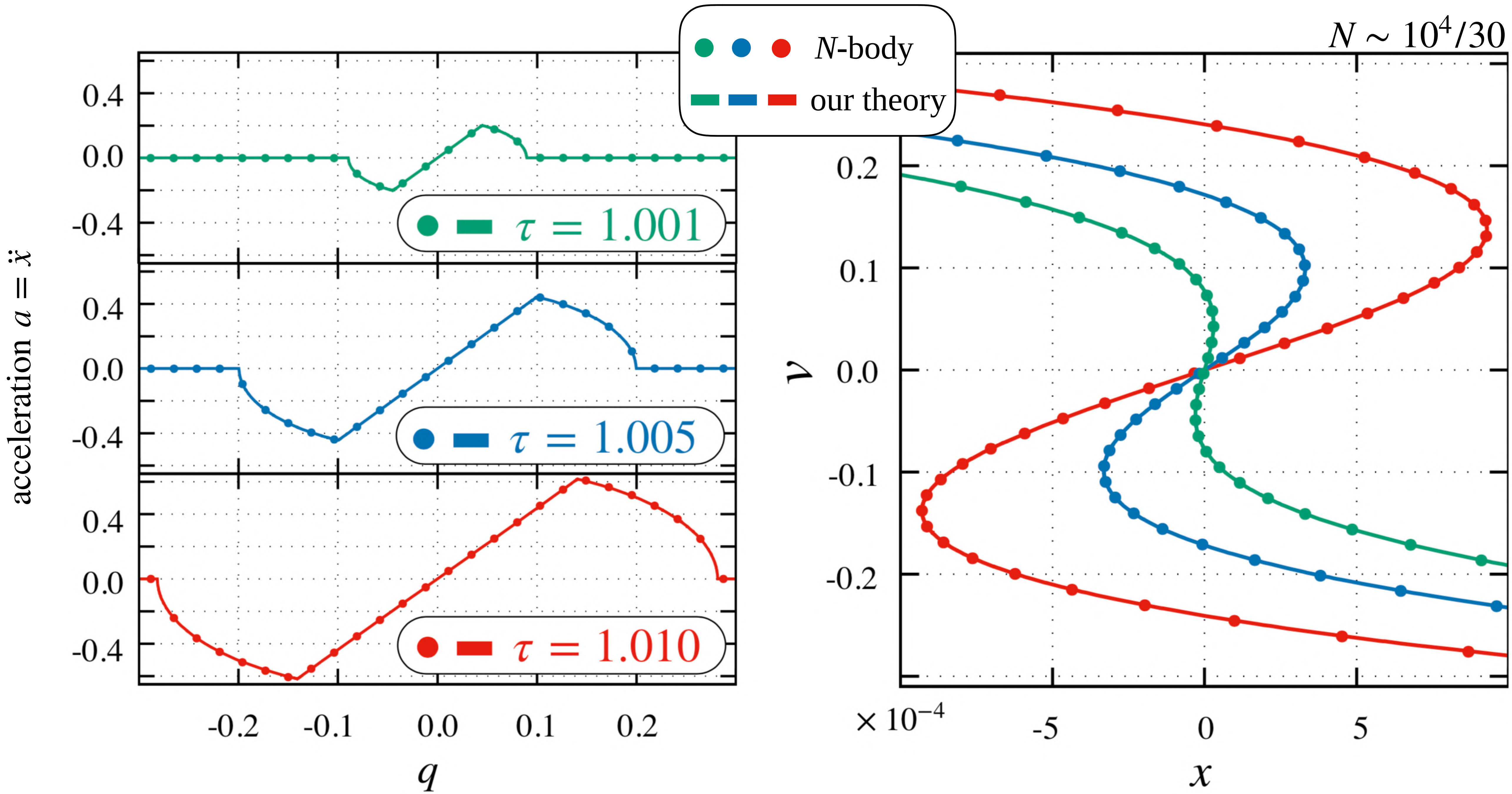


$$\ddot{\boldsymbol{x}}(\boldsymbol{q},a) \propto -\nabla_x \varphi_{\rm g}(\boldsymbol{x}_{\rm sc}(\boldsymbol{q},a))$$

computation requires catastrophe theory

<u>exact</u> until shell-crossing, <u>approximative</u> shortly after

[CR, Frisch & Hahn'19]



Findings in 1D:

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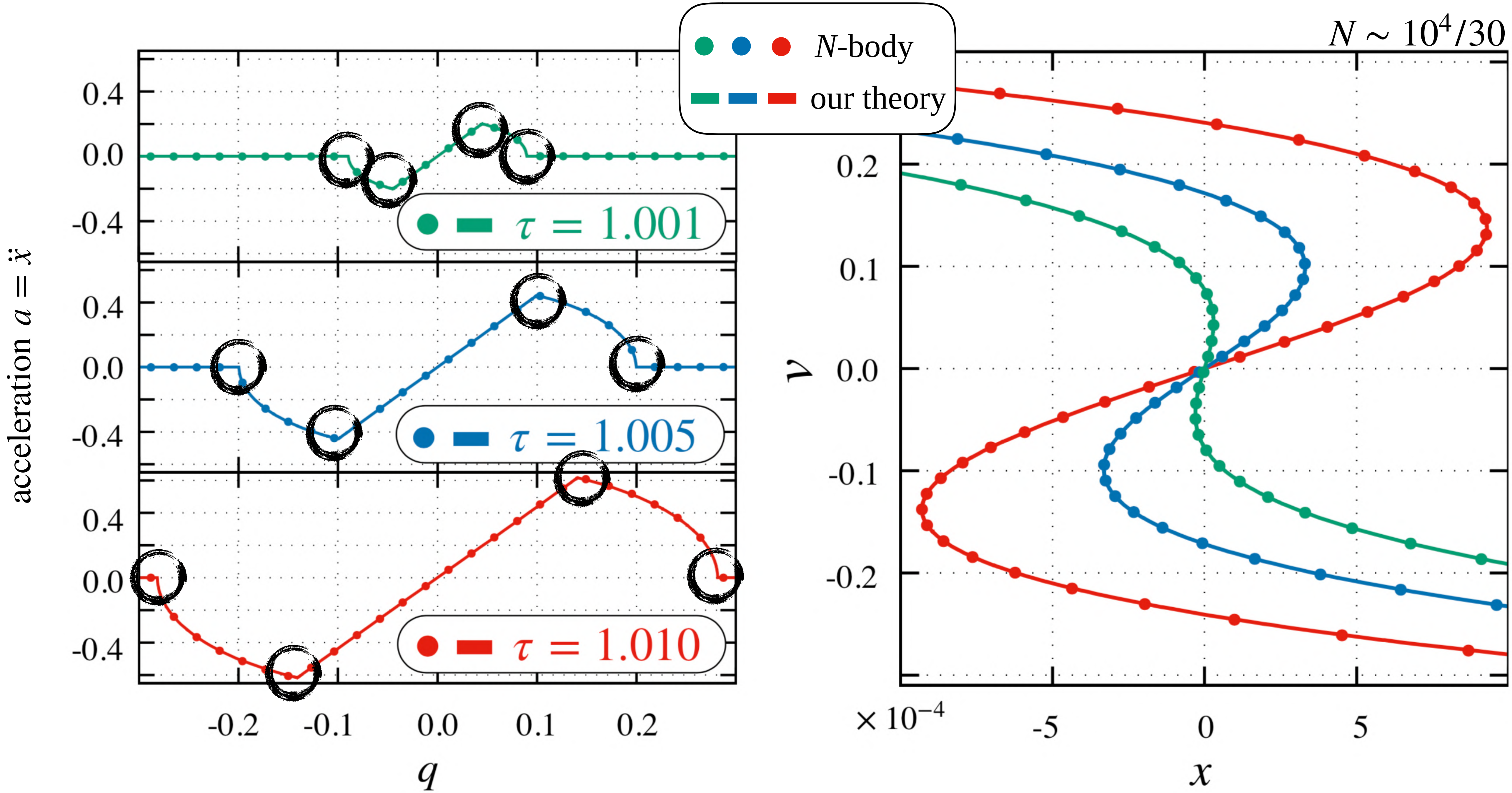
Lab. Lagrange, Nice

[CR, Frisch & Hahn'19]



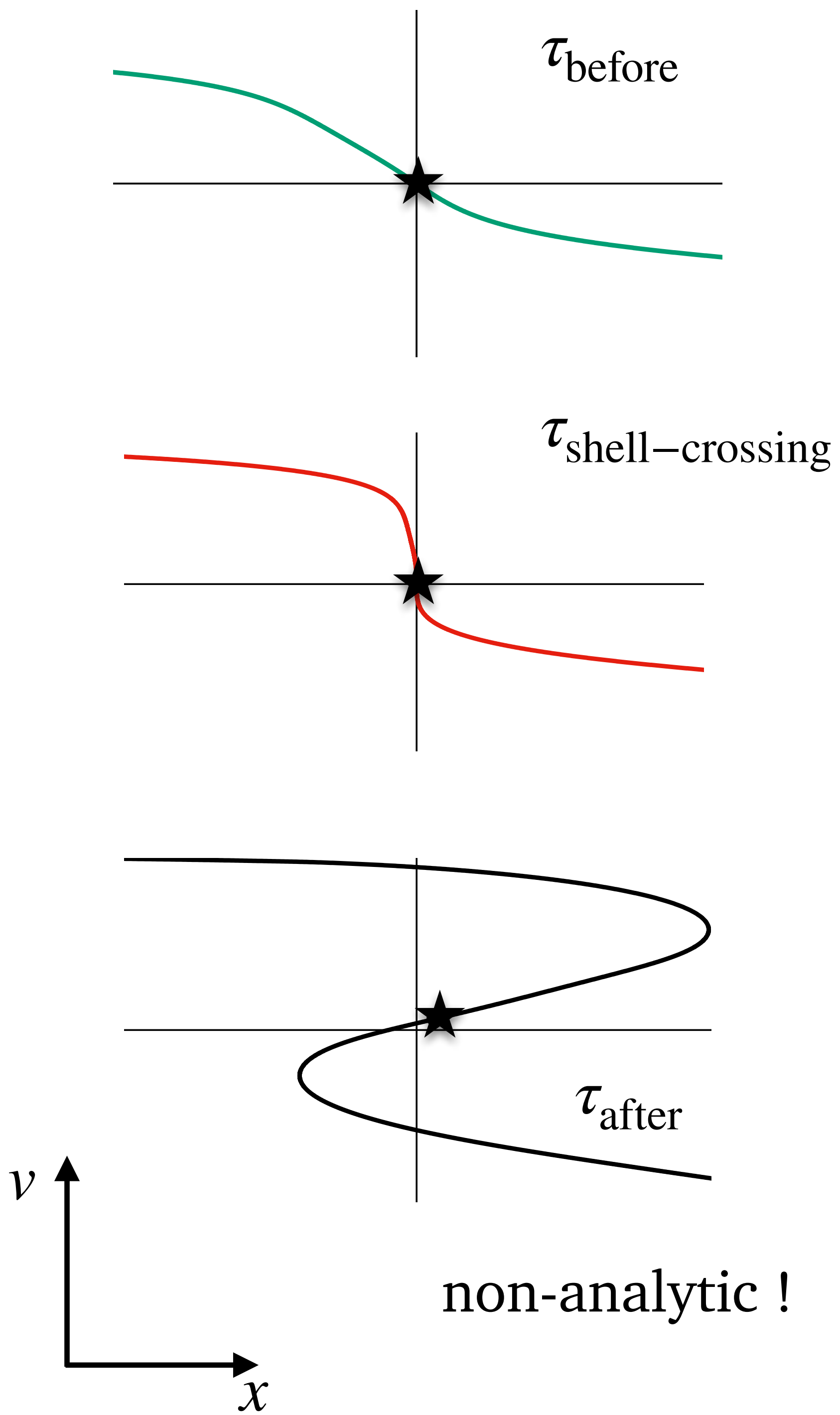
Derivatives in phase-space blow up, evidencing truly singular behaviour in Vlasov-Poisson !

By contrast, the well-known density singularities are not really a problem in phase-space, as they appear as projection effects (there is no 'shell-crossing in 6D').

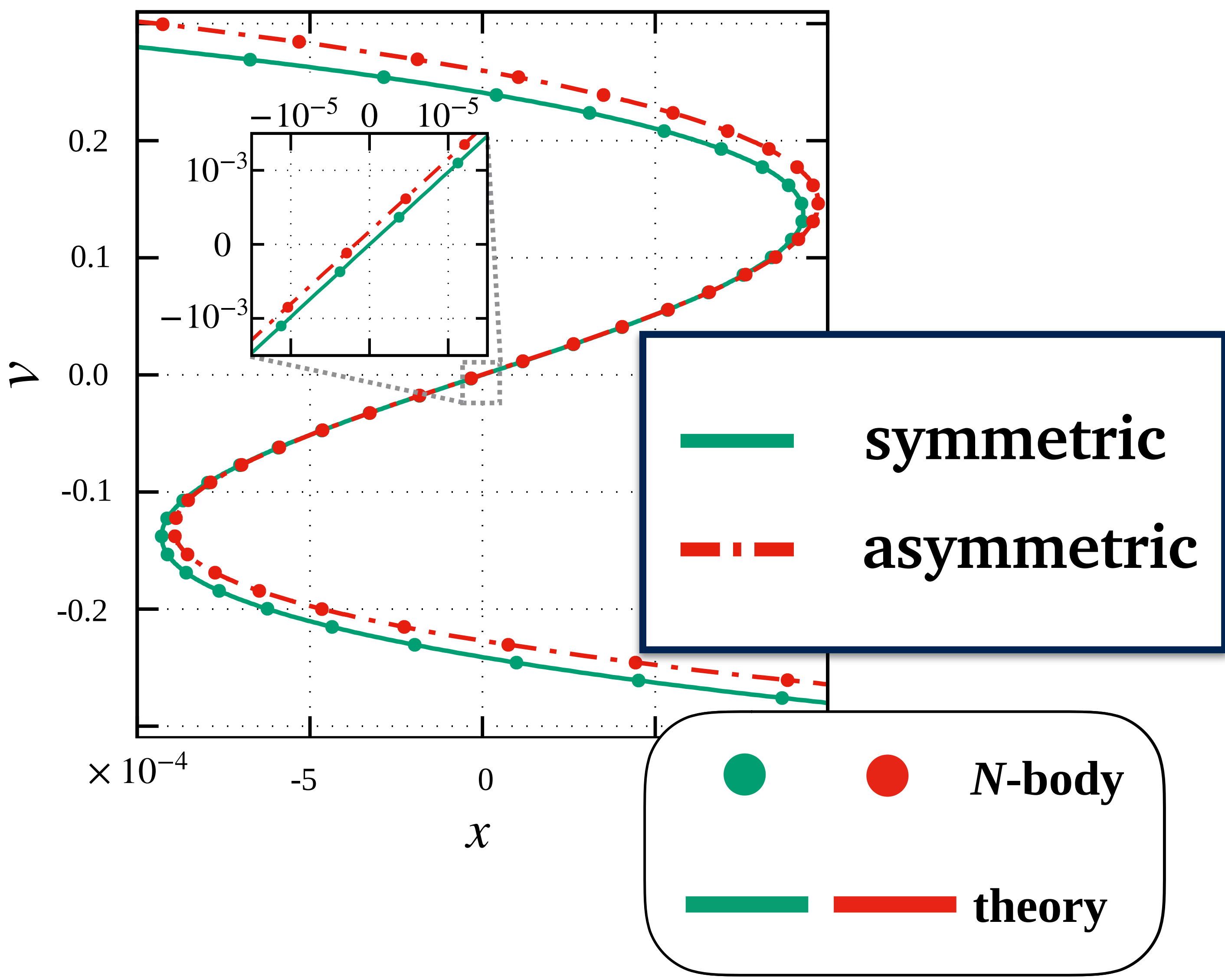
… and another singularity in 1D

[CR,Frisch&Hahn'19]

sudden movement of particle ★ at $q = 0$ due to forcing **asymmetry**, which kicks in only after shell-crossing:



non-analytic !



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[CR,Frisch&Hahn'19]

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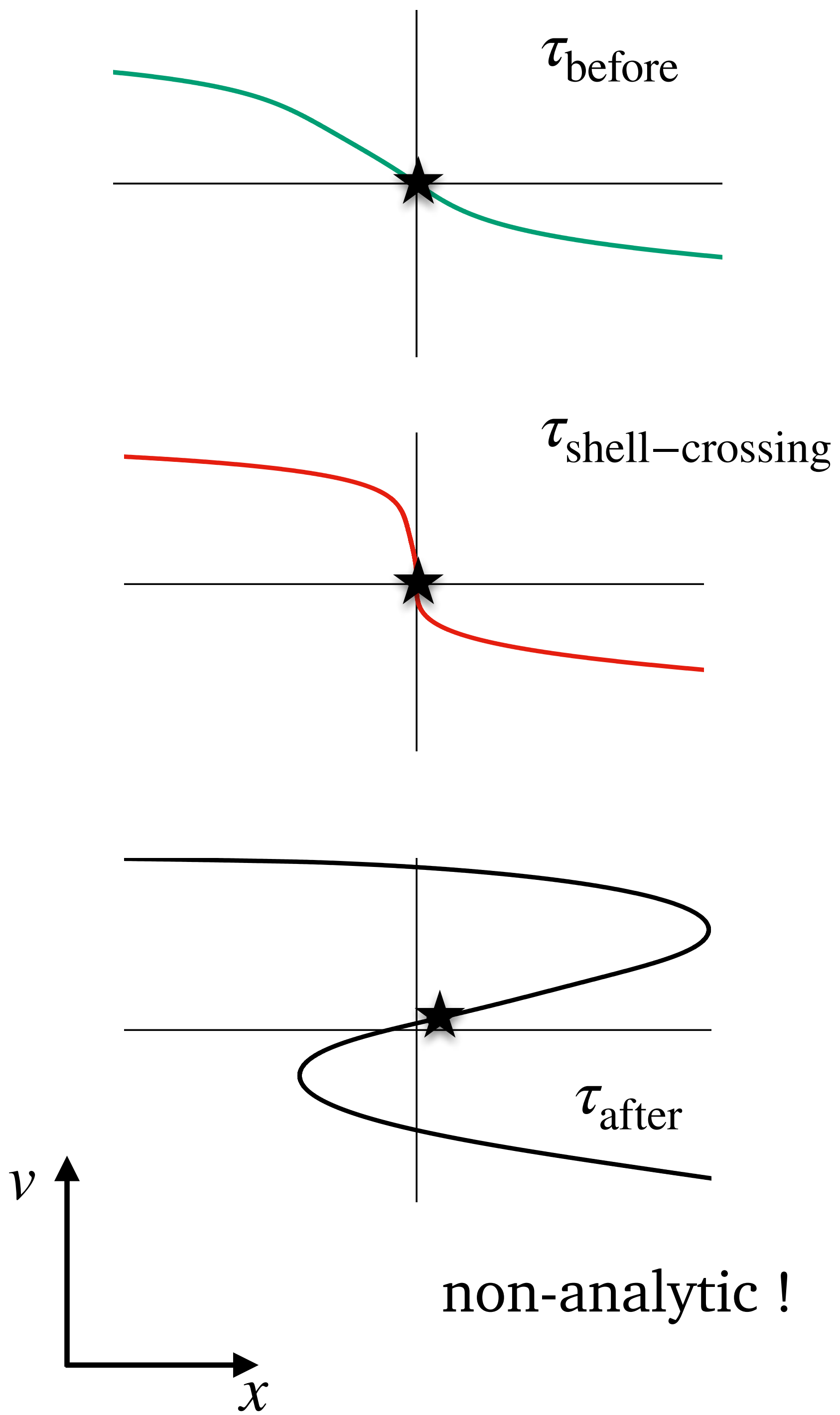

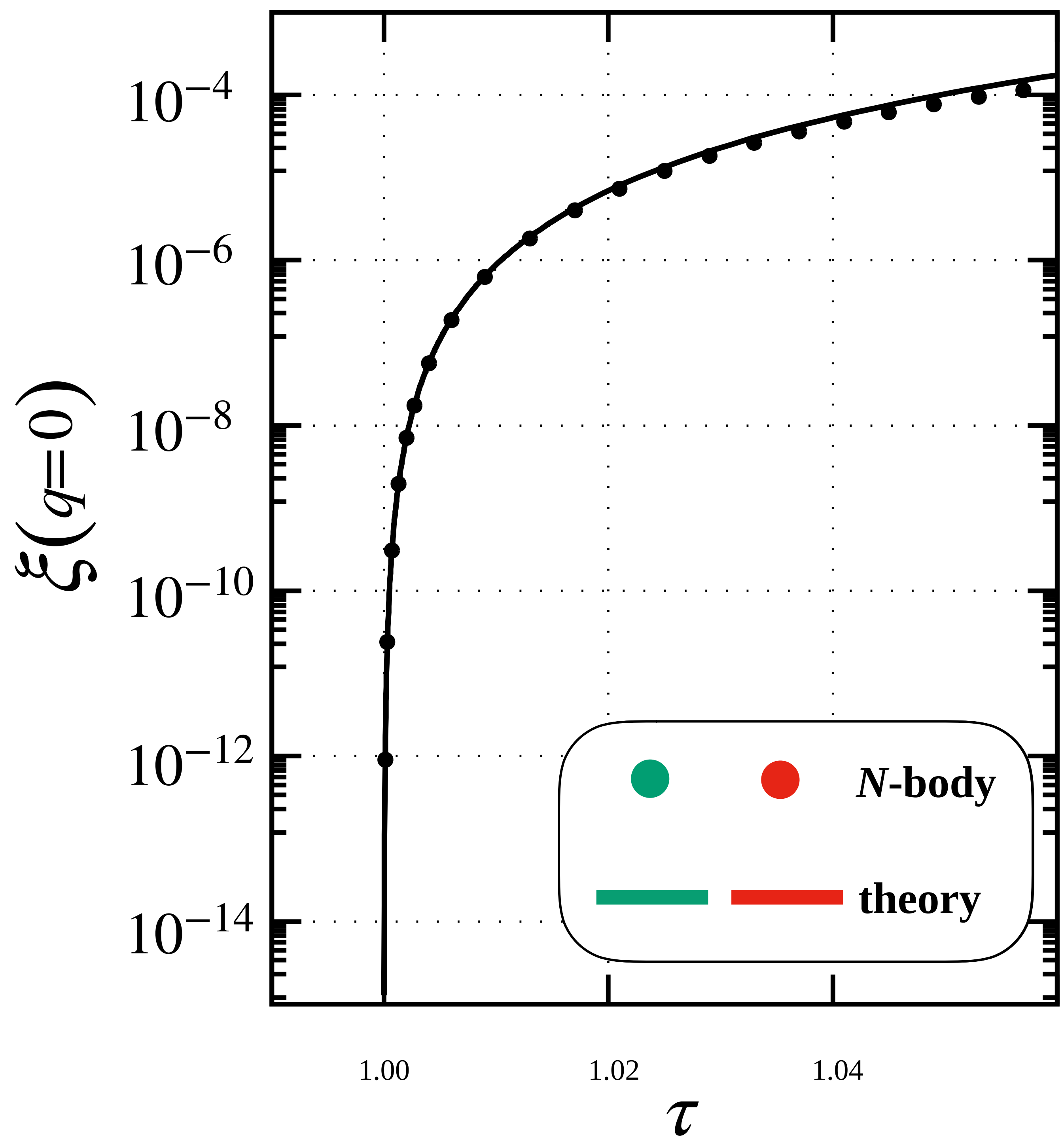

to determine this in theory: exploit invariance of Vlasov-Poisson under non-Galilean transformations

Conclusions & Outlook

- significant theoretical progress on a highly non-linear problem
  (a subject that in cosmology is usually reserved for simulations)
- we have now tools to finally pin down the first shell-crossing
- direct application: setting up ICs for simulations as accurate and late as possible
- indirect applications:
  - **Numerical code for Lagrangian recursion relations:** straightforward to adopt to incompressible Euler in 3D
  - **Post-shell-crossing theory:** could be applied to investigate problems in plasma physics (e.g., bump-on-tail instability, multiple cold or warm beams)